

A Broad Class of Conservative Numerical Methods for Dispersive Wave Equations

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Abstract. We develop a general framework for designing conservative numerical methods based on summation by parts operators and split forms in space, combined with relaxation Runge-Kutta methods in time. We apply this framework to create new classes of fully-discrete conservative methods for several nonlinear dispersive wave equations: Benjamin-Bona-Mahony (BBM), Fornberg-Whitham, Camassa-Holm, Degasperis-Procesi, Holm-Hone, and the BBM-BBM system. These full discretizations conserve all linear invariants and one nonlinear invariant for each system. The spatial semidiscretizations include finite difference, spectral collocation, and both discontinuous and continuous finite element methods. The time discretization is essentially explicit, using relaxation Runge-Kutta methods. We implement some specific schemes from among the derived classes, and demonstrate their favorable properties through numerical tests.

AMS subject classifications: 65M12, 65M70, 65M06, 65M60, 65M20, 35Q35

Key words: Invariant conservation, summation by parts, finite difference methods, Galerkin methods, relaxation schemes.

1 Introduction

In this work we study and develop numerical discretizations for nonlinear dispersive wave equations. One of the most important features of such equations is the existence of nonlinear invariants. In addition to the total mass, many dispersive wave models possess other invariants that may represent the energy or another important physical

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quantity. Perhaps the most interesting feature of these systems, related to the presence of conserved quantities, is the existence of solitary wave solutions. For non-integrable systems, numerical methods are an essential tool for studying solitary waves; even for integrable systems, numerical methods are very useful for exploring solution behavior [7]. Both analysis and numerical experiments have demonstrated that such studies are best undertaken using numerical methods that exactly preserve the invariants of the system in question [8, 35]. Specifically, conservative methods possess discrete solitary wave solutions that accurately approximate the true solitary waves, with an amplitude that is constant in time and a phase error that grows linearly in time [30]. In contrast, non-conservative methods typically yield discrete solutions with amplitude errors that grow linearly in time and (therefore) phase errors that grow quadratically in time. Conservative methods are thus especially desirable for conducting studies of solitary wave properties such as speed-amplitude relationships and solitary wave interactions [35], and for long-time simulations. At the same time, discrete conservation properties can be useful for proving numerical stability.

Significant work has been devoted to the development of conservative methods for certain nonlinear dispersive wave equations [13, 19, 20, 33–36, 92, 93, 103, 110–112]. Nevertheless, and despite their known advantages, conservative fully-discrete schemes are not widely available for many important dispersive nonlinear wave equations, and most methods being proposed and used are non-conservative; see e.g. [4, 7, 14, 18, 37–39, 76, 99, 100, 104]. Indeed, the development of accurate and stable schemes (even without nonlinear invariant conservation) is a challenging task and often requires the application of implicit time discretizations [17, 38, 109].

Usually, proving the conservation of invariants of dispersive partial differential equations (PDEs) at the continuous level requires application of the product/chain rule and integration by parts. To mimic this procedure at the semidiscrete level (discrete in space, continuous in time), summation by parts (SBP) operators are used, which provide a discrete analogue of integration by parts. A review of the relevant theory can be found in [28, 41, 95]. Nowadays, many different schemes have been formulated in the SBP framework, e.g. finite difference [94], finite volume [73, 74], discontinuous Galerkin [45], and flux reconstruction methods [87]. At internal interfaces or external boundaries, SBP methods can be combined with a weak imposition of interface/boundary conditions using so-called simultaneous approximation terms (SATs) to bound the energy of the semidiscretization [23, 24].

Since the chain and product rules cannot hold discretely for many high-order discretizations [82], split forms that preserve local conservation laws are used; cf. [43]. These are related to entropy-conservative methods in the sense of Tadmor [42, 60, 79, 96]. Although the idea to use split forms is not exactly new [90, eq. (6.40)], it is still state of the art and enables the construction of numerical methods with desirable properties [46]. Conservative discretizations based on classical finite element methods require the exact integration of nonlinear terms, which can become very costly or even impossible for non-polynomial nonlinearities. Conservative methods based on split forms do not require ex-