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Arbitrarily High-Order (Weighted) Essentially Non-Oscillatory Finite Difference Schemes for Anelastic Flows on Staggered Meshes

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Abstract. We propose a WENO finite difference scheme to approximate anelastic flows, and scalars advected by them, on staggered grids. In contrast to existing WENO schemes on staggered grids, the proposed scheme is designed to be arbitrarily high-order accurate as it judiciously combines ENO interpolations of velocities with WENO reconstructions of spatial derivatives. A set of numerical experiments are presented to demonstrate the increase in accuracy and robustness with the proposed scheme, when compared to existing WENO schemes and state-of-the-art central finite difference schemes.

AMS subject classifications: 65M06, 65M22, 76M20

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1 Introduction

In numerical modeling of fluid systems that are characterized by a low Mach number, it is often advantageous to introduce approximations to the fully compressible equations that eliminate acoustic waves. Doing so allows the explicit time integration of the governing equations to take much longer time steps in comparison to a similar integration of the compressible equations. This is because numerical stability in the soundproofed system does not depend on the phase velocity of acoustic waves. Various approximations to the compressible equations that eliminate acoustic waves have been developed including the

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incompressible, Boussinesq, and pseudo-incompressible approximations. The choice of which approximation is used is determined by the properties of the system being studied.

In atmospheric science, the anelastic system of equations is widely used as the basis for limited area models, because many atmospheric flows are buoyancy driven, and the stratification of the atmosphere makes it natural to assume that vertical gradients of density are much larger than horizontal gradients. In the anelastic system, the momentum, entropy, and continuity equations are given by

$$\frac{\partial U}{\partial t}+\frac{1}{\rho_0}\nabla\cdot(\rho_0 U\otimes U)=-\nabla\left(\frac{p'}{\rho_0}\right)+be_3, \tag{1.1}$$

$$\frac{\partial s}{\partial t}+\frac{1}{\rho_0}\nabla\cdot(\rho_0 Us)=0, \tag{1.2}$$

$$\nabla\cdot(\rho_0 U)=0, \tag{1.3}$$

respectively. Here $U=(u_1,u_2,u_3)\in\mathbb{R}^3$ is the fluid velocity, ρ_0 is a horizontally homogeneous reference state density, $p'=p-p_0$ is the dynamic pressure, b is the buoyancy, $e_3=(0,0,1)$, and s is the specific entropy defined as in [15]. Following [12], we define ρ_0 to be consistent with an isentropic and hydrostatic state and where the buoyancy $b=g(\rho/\rho_0-1)$ couples the momentum to thermodynamics, and ρ is determined from the equation of state. For most of this paper, we focus on Eqs. (1.1) and (1.3), as with this choice, Eq. (1.2) is only coupled to the other two only through $b=b(\rho(s))$, and can be treated as a source term.

Given their importance in applications, a large variety of numerical methods for approximating anelastic (incompressible) flows are available. In applications where complex domain geometries are rarely encountered, finite difference schemes are a popular discretization framework as they can account for more general boundary conditions than spectral methods [3]. Alternative numerical frameworks, such as the finite element method, discontinuous Galerkin methods, etc. are possible approaches for these problems, particularly on domains with complex geometries. However, on account of their computational efficiency and simplicity of implementation, finite difference methods remain a very attractive option for many applications, particularly in climate and weather modelling.

The most straightforward finite difference discretizations of (1.1) and (1.3) are on *collocated* meshes, where one evolves point values of the velocity and pressure at cell centers. However, it is well known that this procedure can lead to what is termed as *velocity-pressure decoupling*, [11] and references therein. Consequently, nonphysical checkerboard modes are obtained as solutions to the (discrete) elliptic equation that one has to solve in order to compute the pressure from the continuity equation (1.3).

A possible remedy for these nonphysical numerical artifacts is the use of *staggered meshes*. In this framework, each velocity component is discretized on the center of the underlying normal cell edge. The pressure (and any passively or actively advected scalars) is discretized at cell centers. The spatial derivatives in (1.1) can then be discretized by