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Fast One-Dimensional Convolution with General Kernels Using Sum-of-Exponential Approximation

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Abstract. Based on the recently-developed sum-of-exponential (SOE) approximation, in this article, we propose a fast algorithm to evaluate the one-dimensional convolution potential $\phi(x)=K*\rho=\int_0^1 K(x-y)\rho(y)dy$ at (non)uniformly distributed target grid points $\{x_i\}_{i=1}^M$, where the kernel $K(x)$ might be singular at the origin and the source density function $\rho(x)$ is given on a source grid $\{y_j\}_{j=1}^N$ which can be different from the target grid. It achieves an optimal accuracy, inherited from the interpolation of the density $\rho(x)$, within $\mathcal{O}(M+N)$ operations. Using the kernel's SOE approximation $K_{\rm ES}$, the potential is split into two integrals: the exponential convolution $\phi_{\rm ES}=K_{\rm ES}*\rho$ and the local correction integral $\phi_{\rm cor}=(K-K_{\rm ES})*\rho$. The exponential convolution is evaluated via the recurrence formula that is typical of the exponential function. The local correction integral is restricted to a small neighborhood of the target point where the kernel singularity is considered. Rigorous estimates of the optimal accuracy are provided. The algorithm is ideal for parallelization and favors easy extensions to complicated kernels. Extensive numerical results for different kernels are presented.

AMS subject classifications: 68Q25, 65R20, 65D15, 65T50

Key words: One dimensional convolution, sum of exponentials, singular kernel, discrete density.

1 Introduction

Pairwise interaction is common and important in computational physics and practical engineering, and it is usually long-ranged and described by a continuous/discrete convolution. For example, the electrostatic interactions of charge carriers are essential in simulating lightning, blue jet/gigantic jet in atmospheric science, or the corona discharges

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around power transmission lines [1]. Such interactions are originally three dimensional and the evaluations will cost huge amount of computational resources. Reduction to lower dimensional convolutions is common and necessary for meaningful long-time simulations [2,3]. Here we focus on the following rescaled one-dimensional convolution

$$\phi(x)=\int_0^1 K(x-y)\rho(y)dy, \quad \forall\, x\in[0,1], \tag{1.1}$$

where the kernel $K(x), x\in[-1,1]$ might be singular at the origin and $\rho(x)$ is the source density. Numerically, the density $\rho(x)$ may be given on a discrete *source* grid of N points, denoted as $\mathcal{S}:=\{y_j\}_{j=0}^N$ which might be distributed nonuniformly, and the *target* grid of M points, denoted as $\mathcal{T}:=\{x_i\}_{i=0}^M$, *does not* necessarily coincide with the source grid. Usually, the source grid is given as finite difference/element/volume grid and is often nonuniform, so is the target grid. In this paper, we aim to design an accurate and fast algorithm for (1.1) on general grids.

When both the source and target grids are uniformly distributed, the evaluation (1.1) boils down to a discrete convolution [4–6] and can be accelerated via the discrete Fast Fourier Transform within $\mathcal{O}(N\log N)$ operations. However, on nonuniform grids, a simple direct summation of the resulted quadrature costs $\mathcal{O}(MN)$ operations, and usually bottlenecks practical simulations, therefore, it is imperative to design fast algorithms for better efficiency while maintaining the accuracy. There have been several work dealing with such convolutions. In 1999, Yarvin and Rokhlin proposed an improved Fast Multipole Method [7] for the one dimensional discrete convolution with singular kernels, including x^{-1}, $\log(x)$ and $x^{-1/2}$. Subsequently, Beylkin [8–10] designed a fast discrete convolution algorithm of complexity $\mathcal{O}(NQ)$, where Q is the number of exponentials, for a wider range of kernels. Recently, Greengard et al. [11] proposed an algorithm for finding nearly optimal SOE approximation for non-oscillatory functions, and applied the SOE to calculate high dimensional spatial volume convolutions. Similar ideas using the recurrence scheme of the exponential function has been developed and applied to many problems in various fields, e.g., the temporal convolution whose integration domain is $[0,t]$ rather than the whole interval $[0,T]$, in the context of nonreflecting boundary condition of the Schrödinger equation, wave equation, and fractional temporal derivatives (see, for example [12–14]).

To compute the potential ϕ, we first construct a sum-of-exponential approximation of the kernel

$$K_{\rm ES}(x):=\sum_{q=1}^{Q}\omega_q e^{-\alpha_q x}, \quad \text{with } \omega_q,\, \alpha_q\in\mathbb{C}, \quad \Re(\alpha_q)>0, \tag{1.2}$$

such that

$$\|K(x)-K_{\rm ES}(x)\|_\infty\le\varepsilon, \quad \forall\, x\in[\delta,1], \tag{1.3}$$

with a small truncation parameter $0<\delta<1$ and a prescribed accuracy $0<\varepsilon\ll 1$, which can be controlled as small as possible, e.g., $10^{-14}\sim 10^{-12}$. The number of exponentials