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An Augmented Lagrangian Uzawa Iterative Method for Solving Double Saddle-Point Systems with Semidefinite $(2,2)$ Block and its Application to DLM/FD Method for Elliptic Interface Problems

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Abstract. In this paper, an augmented Lagrangian Uzawa iterative method is developed and analyzed for solving a class of double saddle-point systems with semidefinite (2,2) block. Convergence of the iterative method is proved under the assumption that the double saddle-point problem exists a unique solution. An application of the iterative method to the double saddle-point systems arising from the distributed Lagrange multiplier/fictitious domain (DLM/FD) finite element method for solving elliptic interface problems is also presented, in which the existence and uniqueness of the double saddle-point system is guaranteed by the analysis of the DLM/FD finite element method. Numerical experiments are conducted to validate the theoretical results and to study the performance of the proposed iterative method.

AMS subject classifications: 65F10, 65F50, 65N30, 65N85

Key words: Double saddle-point problem, augmented Lagrangian Uzawa method, elliptic interface problem, distributed Lagrange multiplier/fictitious domain (DLM/FD) method.

1 Introduction

In this paper, we study a type of augmented Lagrangian Uzawa iterative method for solving a large-scale sparse linear algebraic system as shown below

$$\mathcal{A}\boldsymbol{u} \equiv \begin{pmatrix} A & 0 & C^T \\ 0 & A_2 & B^T \\ C & B & 0 \end{pmatrix} \begin{pmatrix} u \\ u_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} F \\ G \\ 0 \end{pmatrix} \equiv \boldsymbol{b}, \tag{1.1}$$

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where, $\mathcal{A}\in\mathbb{R}^{(n+2m)\times(n+2m)}$ is the coefficient matrix, and the right hand side $\boldsymbol{b}\in\mathbb{R}^{n+2m}$. Inside the coefficient matrix $\mathcal{A}$, the block $A\in\mathbb{R}^{n\times n}$ is symmetric positive definite (SPD), $A_2\in\mathbb{R}^{m\times m}$ is symmetric positive semidefinite (SPS), $B\in\mathbb{R}^{m\times m}$ is invertible and $C\in\mathbb{R}^{m\times n}$ with $n>m$. Such linear systems can be derived from the DLM/FD finite element discretization for elliptic interface problems [1, 7] and parabolic interface problems [24], where the distributed Lagrange multiplier is employed and acts as a source term for both unknown quantities u and u_2 in two overlapping domains. We remark that similar, and even much more complicated, multiple saddle-point systems can also be generated from the DLM/FD finite element method for Stokes- [16, 17], Stokes/elliptic- [22] and Stokes/parabolic [23] interface problems, moreover, for fluid-structure interaction (FSI) problems [11,12,28].

Generally, the linear system (1.1) can be viewed as a standard saddle-point system, if we split its coefficient matrix as the following 2×2 block matrix

$$\mathcal{A}=\left(\begin{array}{cc|c} A & 0 & C^T \\ 0 & A_2 & B^T \\ \hline C & B & 0 \end{array}\right). \tag{1.2}$$

Among the iterative method for solving the saddle-point systems, the Uzawa method, augmented Lagrangian method and their variants are very popular and widely used, due to their simplicity, the minimum requirement of computer memory and the parallel efficiency on emerging multicore and hybrid architectures. The reader are referred to [2, 3, 6, 8–10, 29, 30] and the references therein. In most of these papers, the convergence analysis are performed under the assumption that the upper left $(1,1)$-block of $\mathcal{A}$ in (1.2) is invertible. However, since the block A_2 in (1.1) is only positive semidefinite, this assumption is not satisfied. Thus the theocratical results therein cannot guarantee the convergence of these Uzawa type iterative methods for solving the linear system (1.1). Note that in [19] an augmented Lagrangian method has be used for solving the saddle-point system with singular or semidefinite upper left $(1,1)$-block of $\mathcal{A}$ in (1.2), but the convergence analysis was not given.

On the other hand, we can also split the coefficient matrix $\mathcal{A}$ as another 2×2 block matrix

$$\mathcal{A}=\left(\begin{array}{c|cc} A & 0 & C^T \\ \hline 0 & A_2 & B^T \\ C & B & 0 \end{array}\right). \tag{1.3}$$

Since the lower right (2,2)-block in (1.3) itself also owns a saddle-point structure, the linear algebraic system (1.1) is thus treated as a class of double saddle-point system, and fits the definition of a multiple saddle-point operator as given in [21]. Recently, there have been several literatures on the iterative method for solving such three-by-three block systems where the double saddle-point structure, instead of the single saddle-point structure, is studied and used for the construction of iterative method in order to reduce the