Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2019-0204

Vol. 30, No. 1, pp. 288-320
July 2021

A High-Order Discontinuous Galerkin Solver for Helically Symmetric Flows

Dominik Dierkes[1,*], Florian Kummer[1] and Dominik Plümacher[1]

[1] *Department of Mechanical Engineering, Technische Universität Darmstadt, Otto-Berndt-Straße 2, 64287 Darmstadt, Germany.*

Received 21 November 2019; Accepted (in revised version) 12 January 2021

Abstract. We present a high-order discontinuous Galerkin (DG) scheme to solve the system of helically symmetric Navier-Stokes equations which are discussed in [28]. In particular, we discretize the helically reduced Navier-Stokes equations emerging from a reduction of the independent variables such that the remaining variables are: t, r, ξ with $\xi = az + b\varphi$, where r, φ, z are common cylindrical coordinates and t the time. Beside this, all three velocity components are kept non-zero. A new non-singular coordinate η is introduced which ensures that a mapping of helical solutions into the three-dimensional space is well defined. Using that, periodicity conditions for the helical frame as well as uniqueness conditions at the centerline axis $r=0$ are derived. In the sector near the axis of the computational domain a change of the polynomial basis is implemented such that all physical quantities are uniquely defined at the centerline.

For the temporal integration, we present a semi-explicit scheme of third order where the full spatial operator is splitted into a Stokes operator which is discretized implicitly and an operator for the nonlinear terms which is treated explicitly. Computations are conducted for a cylindrical shell, excluding the centerline axis, and for the full cylindrical domain, where the centerline is included. In all cases we obtain the convergence rates of order $\mathcal{O}(h^{k+1})$ that are expected from DG theory.

In addition to the first DG discretization of the system of helically invariant Navier-Stokes equations, the treatment of the central axis, the resulting reduction of the DG space, and the simultaneous use of a semi-explicit time stepper are of particular novelty.

AMS subject classifications: 76T99, 35J75

Key words: Discontinuous Galerkin (DG), helical flows.

*Corresponding author. *Email addresses:* dierkes@fdy.tu-darmstadt.de (D. Dierkes), kummer@fdy.tu-darmstadt.de (F. Kummer), pluemacher@fdy.tu-darmstadt.de (D. Plümacher)

1 Introduction

Helical flow structures appear in various natural phenomena and technological devices, for example, in the wake of windmills [39], as wing tip vortices [32], in astrophysical plasmas (see e.g. [4]) and in laboratory applications, including "straight tokamak" plasma flow approximations, (see e.g. [27, 35]) and other experiments. In particular, helical vortex structures were observed by [34] in experiments with swirling flows in a cylindrical tube, and as such, they are part of the various flow structures observed in the known vortex breakdown.

Various groups have worked on the theoretical description of helical flows in recent decades. The simplest approach here is to introduce a helical coordinate $\xi = az+b\varphi$, $a,\ b = \text{const.} \neq 0$ and to assume that all physical quantities depend on the cylinder radius r and the helical coordinate ξ. Helically invariant flows include translationally and axially invariant ones as special cases. For both steady Euler equations describing incompressible fluid flows and for plasma equilibrium equations in the magnetohydrodynamics (MHD) framework, the helical invariance requirement allows to reduce the governing equations to a single partial differential equation (PDE) known as the JFKO equation [27]. This important equation generalizes the famous Bragg-Hawthorne-Grad-Rubin-Shafranov equation [7, 23, 36] describing steady axisymmetric inviscid flows onto the helically invariant case. Families of exact solutions of JFKO equations are known, including those derived by [6] (see also [5, 10]). In the more general context of helical geometry, several works focused on twisted pipes following a given spatial curve (see [19, 20, 38, 41]). Using non-orthogonal and local-orthogonal coordinate systems, the effects of pipe curvature and torsion on the flow were investigated. Special analytical solutions of steady flows in helically symmetric pipes were found by [43]. In [12] a DNS code for the helical invariant Navier-Stokes equations in a generalized vorticity-streamfunction formulation has been developed. In [16] the three-dimensional Euler equations are reduced to a linear equation, assuming that the flow has helical symmetry and consists of a rigidly rotating basic part and a Beltrami disturbance part. Further, the authors derived exact solutions for flows in a straight pipe of circular cross section. Exact solutions for helical flows of a Maxwell fluid constrained between two infinite coaxial circular cylinders were derived by [26]. The present introduction as well as additional results on helical flows can be found in [14].

The full three dimensional system of incompressible constant-density Euler- and Navier-Stokes equations under the assumption of helical symmetry have been derived and analyzed in [28]. In particular, various new conservation laws admitted by the model have been found in primitive variables and using the vorticity formulation. A general helically symmetric setting was used, where all three velocity components and the pressure are generally nonzero, and may depend on the time t, the cylindrical radius r, and the helical variable

$$\xi = az+b\varphi. \tag{1.1}$$