

On False Accuracy Verification of UMUSCL Scheme

Hiroaki Nishikawa*

National Institute of Aerospace, 100 Exploration Way, Hampton, VA 23666, USA.

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Abstract. In this paper, we reveal a mechanism behind a false accuracy verification encountered with unstructured-grid schemes based on solution reconstruction such as UMUSCL. Third- (or higher-) order of accuracy has been reported for the Euler equations in the literature, but UMUSCL is actually second-order accurate at best for nonlinear equations. False high-order convergence occurs generally for a scheme that is high order for linear equations but second-order for nonlinear equations. It is caused by unexpected linearization of a target nonlinear equation due to too small of a perturbation added to an exact solution used for accuracy verification. To clarify the mechanism, we begin with a proof that the UMUSCL scheme is third-order accurate only for linear equations. Then, we derive a condition under which the third-order truncation error dominates the second-order error and demonstrate it numerically for Burgers' equation. Similar results are shown for the Euler equations, which disprove some accuracy verification results in the literature. To be genuinely third-order, UMUSCL must be implemented with flux reconstruction.

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Key words: UMUSCL, high-order, finite-volume, finite-difference, unstructured grids, reconstruction schemes.

1 Introduction

This paper is a sequel to the two previous papers [1,2], where we clarified the MUSCL and QUICK schemes towards the clarification of economical high-order unstructured-grid schemes for practical computational fluid dynamics (CFD) solvers, e.g., third-order UMUSCL with $\kappa = 1/2$ [3], $\kappa = 1/3$ [4,5], or $\kappa = 0$ [6,7]. In this paper, we will clarify one more confusion: the false accuracy verification of the UMUSCL scheme.

The UMUSCL scheme of Burg [3] is generally considered as an unstructured-grid extension of Van Leer's κ -reconstruction scheme [4,5] and has been widely employed in practical CFD solvers [8–20] with a confusion over the value of κ for giving third-order accuracy on regular or one-dimensional grids: $\kappa = 1/2$ [3], $\kappa = 1/3$ [4,5], or $\kappa = -1/6$

*Corresponding author. *Email address:* hiro@nianet.org (H. Nishikawa)

[10, 13]. The confusion arises mainly from taking different combinations of numerical solution and discretization types : numerical solutions stored as point-valued or cell-averaged solutions; discretizations of a differential form of a conservation law at a point (finite-difference) or of an integral form over a cell (finite-volume). For example, Burg originally proposed the UMUSCL scheme as a finite-volume scheme with point-valued numerical solutions stored at nodes on an unstructured grid [3]. Note that we know he used point-valued numerical solutions because that is the only way he could obtain third-order accuracy with $\kappa = 1/2$ for a one-dimensional nonlinear system (see Refs. [1, 2] for details). This is already confusing because the MUSCL scheme is based on cell-averaged numerical solutions, not point-valued solutions (see Ref. [1]). As clarified in the previous paper [2], a third-order finite-volume scheme with point-valued numerical solutions is nothing but the QUICK scheme and therefore the UMUSCL scheme should have been called the UQUICK scheme. In fact, third-order accuracy with $\kappa = 1/2$, which is true for the QUICK scheme, has been confirmed for a one-dimensional steady conservation law by Burg [3] (see also Ref. [2]). Third-order accuracy demonstrated by Burg is genuine but only for one-dimensional problems; it cannot be third-order in multi-dimensions even for Cartesian grids unless the flux is integrated over a face by a high-order quadrature formula.

To be even more confusing, in many or perhaps all practical unstructured-grid codes, the UMUSCL scheme is implemented not as a finite-volume scheme but as a point-wise scheme with the time derivative and source/forcing terms evaluated at a solution point (a node or a cell center) [10, 11, 13–15, 18]. In this paper, we will focus on this particular implementation. We will call it simply the UMUSCL scheme but it should not be confused with Burg's finite-volume UMUSCL scheme. Then, the fact that the scheme has been shown to achieve up to fourth-order accuracy with a single flux evaluation per face on Cartesian grids [10, 13–15] indicates that the scheme is actually a finite-difference scheme, approximating the differential form of a target equation at a solution point. Note that it does not matter how the discretization is derived; the resulting discretization must be high-order as a finite-difference scheme, not as a finite-volume scheme because the time derivative and source/forcing terms are not integrated with high-order quadrature over a cell. Therefore, the UMUSCL scheme corresponds to neither the MUSCL scheme nor the QUICK scheme and does not achieve high-order accuracy in the same way as the MUSCL scheme does. As we will show, the UMUSCL scheme is third-order accurate for linear equations with $\kappa = 1/3$, but only second-order accurate when applied to nonlinear conservation laws. This feature is common to conservative finite-difference schemes with a flux evaluated with reconstructed solutions as in MUSCL (e.g., those in Refs. [21, 22]). Hence, high-order verification results reported in the literature for the UMUSCL scheme applied to the Euler equations are misleading and/or misinterpreted. To be genuinely third- or higher-order accurate, it is necessary to directly reconstruct the flux as pointed out for somewhat similar schemes in Ref. [23, 24]. In other words, the UMUSCL scheme of Refs. [10, 13, 14] can be easily made third- or higher-order by direct flux reconstruction.

It is worth pointing out that a similar unstructured-grid MUSCL scheme had already