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New Unconditionally Stable Schemes for the Navier-Stokes Equations

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Abstract. In this paper we propose some efficient schemes for the Navier-Stokes equations. The proposed schemes are constructed based on an auxiliary variable reformulation of the underlying equations, recently introduced by Li et al. [20]. Our objective is to construct and analyze improved schemes, which overcome some of the shortcomings of the existing schemes. In particular, our new schemes have the capability to capture steady solutions for large Reynolds numbers and time step sizes, while keeping the error analysis available. The novelty of our method is twofold: i) Use the Uzawa algorithm to decouple the pressure and the velocity. This is to replace the pressure-correction method considered in [20]. ii) Inspired by the paper [21], we modify the algorithm using an ingredient to capture stationary solutions. In all cases we analyze a first- and second-order schemes and prove the unconditionally energy stability. We also provide an error analysis for the first-order scheme. Finally we validate our schemes by performing simulations of the Kovasznay flow and double lid driven cavity flow. These flow simulations at high Reynolds numbers demonstrate the robustness and efficiency of the proposed schemes.

AMS subject classifications: 65N30, 35Q30, 35A35, 65N22

Key words: Navier-Stokes equations, auxiliary variable approach, unconditional stability, finite element method.

1 Introduction

The numerical resolution of Navier-Stokes equations (NSE) for incompressible fluids requires solving several issues, especially in the case of unsteady flows in high Reynolds.

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Among these challenges we can quote two important ones: 1) the treatment of the coupling of the velocity and pressure due to the incompressibility constraint; 2) the treatment of the convection term.

There exist many classical methods to deal with the coupling of the velocity and pressure in the Navier-Stokes equations. The most common one is certainly the projection-type methods, either pressure correction or velocity correction, see, e.g, [3, 15, 27, 35, 37] and the references therein. The so-called Uzawa algorithm is also an efficient way to decouple the velocity and pressure [5–7, 23–25, 34, 38]. It employs a block Gauss elimination in the discrete saddle-point problem to decouple the velocity from the pressure and yields two positive definite symmetric systems. The Uzawa approach avoids the side effects of numerical boundary layers, which may be caused by time-splitting and responsible for the precision limitation of the pressure. However, the classical Uzawa system suffers from the inverse of the Helmholtz equations involved in the pressure system. To overcome this, Fortin et al. [7] presented some iteration methods to solve the NSE. Chen et al. [5] proposed a Uzawa method based on the mixed finite element method to solve the steady NSE. Recently, Shu et al. [34] developed a local and parallel Uzawa finite element method for the generalized NSE.

In this paper we will adopt the Uzawa algorithm in the framework of the auxiliary variable approach, which has been found to be an efficient tool to construct stable schemes for several types of equations. The motivation will become clear later.

Dealing with the nonlinear terms, there are two points need to be balanced: the larger time step size, and the smaller computational cost. There are numerous works devoted to these points [1, 2, 4, 9, 11, 12, 28, 30, 39, 40]. Some classical methods treat nonlinear terms to be fully-implicit or semi-implicit. Some of them are unconditionally stable, so that large or relatively large time step sizes are allowed. However those schemes may lead to time-dependent coefficient matrices, thus computational cost at each time step can be expensive. An explicit way is usually easier and cheaper in computation, no need to reconstruct coefficient matrix at each time step, but meanwhile the restriction on time step size suffers from stability requirement. In very recent years, the so-called scalar auxiliary variable (SAV) approach becomes a popular way to deal with nonlinear terms. It has been found to be a quite general method applicable to gradient flow models [14, 16, 20–22, 26, 32, 33, 41] and the NSE [20–22]. In SAV, the nonlinear terms are treated to be explicit, the coefficient matrix is time-independent, only a set of constant coefficient equations needs to be solved at each time step. Moreover, the unconditional stability implies that relatively large time step is allowed in long-time simulation.

Even so, there are still some issues concerning these methods, which are worth to consider in the Navier-Stokes approximation:

(i) The current methods in [20, 22] have a perfect stability and accuracy in unsteady flow simulation, even with high Reynolds numbers. However the current methods could not guarantee accuracy with large or relatively large time step in steady flow simulation, especially with high Reynolds numbers. As a fact, it is known that the accuracy should not depend on the time step in steady flow simulation if unconditional stability stands.