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# Three Discontinuous Galerkin Methods for One- and Two-Dimensional Nonlinear Dirac Equations with a Scalar Self-Interaction

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**Abstract.** This paper develops three high-order accurate discontinuous Galerkin (DG) methods for the one-dimensional (1D) and two-dimensional (2D) nonlinear Dirac (NLD) equations with a general scalar self-interaction. They are the Runge-Kutta DG (RKDG) method and the DG methods with the one-stage fourth-order Lax-Wendroff type time discretization (LWDG) and the two-stage fourth-order accurate time discretization (TSDG). The RKDG method uses the spatial DG approximation to discretize the NLD equations and then utilize the explicit multistage high-order Runge-Kutta time discretization for the first-order time derivatives, while the LWDG and TSDG methods, on the contrary, first give the one-stage fourth-order Lax-Wendroff type and the two-stage fourth-order time discretizations of the NLD equations, respectively, and then discretize the first- and higher-order spatial derivatives by using the spatial DG approximation. The $L^2$ stability of the 2D semi-discrete DG approximation is proved in the RKDG methods for a general triangulation, and the computational complexities of three 1D DG methods are estimated. Numerical experiments are conducted to validate the accuracy and the conservation properties of the proposed methods. The interactions of the solitary waves, the standing and travelling waves are investigated numerically and the 2D breathing pattern is observed.

**AMS subject classifications**: 65M60, 35L05, 81Q05, 81-08

**Key words**: Nonlinear Dirac equation, discontinuous Galerkin method, Lax-Wendroff type time discretization, two-stage fourth-order accurate time discretization, Runge-Kutta method, solitary wave interaction.

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# 1 Introduction

The Dirac equation is a relativistic wave equation in particle physics, and provides a natural description of an electron [16]. It predicted the existence of "negative" states for the electron and proton, and thus successfully predicted the existence of antimatter. After Dirac found the linear equation of the electron, the basic idea of nonlinear description of the elementary particle with spin-1/2 appeared, which made it possible to consider its self-interaction [20, 21, 29], and the nonlinear Dirac (NLD) equation was proposed as a possible basis model for a unified field theory [27]. The NLD equation allows solitary wave or particle-like solutions (the stable localized solutions with finite energy and charge) [39], that is to say, the particles appear as intense localized regions of field [50]. Around the 1970s and 1980s, wide interest of physicists and mathematicians was attracted by different NLD models with different self-interactions, mainly including the Thirring model [48], the Soler model [47], the Gross-Neveu model [23] (equivalent to the massless Soler model), and the bag model [36] (i.e. the solitary waves with finite compact support), especially to look for the solitary wave solutions and to investigate the related physical and mathematical properties [39]. Since entering the 21st century, the Dirac equation is used to study the structures and dynamical properties of the two-dimensional (2D) materials such as graphene and graphite [1,8,18,37] and the relativistic effects in molecules in super intense lasers [19] etc. Moreover, the Bose-Einstein condensates in a honeycomb optical lattice can also be described by a NLD equation in the long wavelength, mean field limit [25]. Mathematical interests related to the NLD equation are mainly manifested in deriving the analytical solitary wave solutions, the stability analysis of the NLD solitary waves, the analysis of global well-posedness and numerical methods etc. For the 1D NLD equation (i.e. one space dimension), several analytical solitary wave solutions are derived in literature. For example, the solitary wave solutions of the 1D NLD equation with arbitrary nonlinearity was studied in [12]. However, for the high-dimensional NLD equation, there are no explicit solitary wave solutions [13]. The stability of the solitary waves can be found in [14, 30, 42]. The readers are referred to the review in [52] and references therein.

Numerical method has become one of the important tools to derive the NLD solitary wave solutions, and to investigate their stability and interaction etc. The Crank-Nicolson (CN) scheme was first proposed for the 1D Soler model and used to simulate the interaction dynamics of the NLD solitary waves in [3, 4]. Such interaction dynamics problem was carefully revisited in [43] by utilizing a fourth-order accurate RKDG method [44]. Besides the recovery of the phenomena in [3], several new ones were observed, e.g. collapse in binary and ternary collisions of two-hump NLD solitary waves [43], a long-lived oscillating state formed with an approximate constant frequency in collisions of two standing waves [44], and the full repulsion in binary and ternary collisions of out-of-phase waves [45]. Those numerical results also showed that the two-hump profile could undermine the stability during the scattering of the NLD solitary waves. It is worth noting that the two-hump profile is first pointed out in [43] and later noticed by other researchers.