Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2021-0011

Vol. **30**, No. 4, pp. 1185-1215
October 2021

Two Nonlinear Positivity-Preserving Finite Volume Schemes for Three-Dimensional Heat Conduction Equations on General Polyhedral Meshes

Menghuan Liu[1,*], Shi Shu[1,*], Guangwei Yuan[2,*] and Xiaoqiang Yue[1,*]

[1] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Key Laboratory of Intelligent Computing & Information Processing of Ministry of Education, School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China.*
[2] *Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100088, China.*

Received 20 January 2021; Accepted (in revised version) 6 May 2021

Abstract. In this article we present two types of nonlinear positivity-preserving finite volume (PPFV) schemes for a class of three-dimensional heat conduction equations on general polyhedral meshes. First, we present a new parameter selection strategy on the one-sided flux and establish a nonlinear PPFV scheme based on a two-point flux with higher efficiency. By comparing with the scheme proposed in [H. Xie, X. Xu, C. Zhai, H. Yong, Commun. Comput. Phys. 24 (2018) 1375–1408], our scheme avoids the assumption that the values of auxiliary unknowns are nonnegative, which makes our interpolation formulae suitable to be constructed by existing approaches with high accuracy and well robustness (e.g., the finite element method), thus enhancing the adaptability to distorted meshes with large deformations. Then we derive a linear multi-point flux involving combination coefficients and, via the Patankar trick, obtain another nonlinear PPFV scheme that is concise and easy to implement. The selection strategy of combination coefficients is also provided to improve the convergence behavior of the Picard procedure. Furthermore, the existence and positivity-preserving properties of these two nonlinear PPFV solutions are proved. Numerical experiments with the discontinuous diffusion scalar as well as discontinuous and anisotropic diffusion tensors are given to confirm our theoretical findings and demonstrate that our schemes both can achieve ideal-order accuracy even on severely distorted meshes.

AMS subject classifications: 65M08, 65N55

Key words: General polyhedral mesh, heat conduction equation, nonlinearity, positivity-preserving, existence.

---

*Corresponding author. *Email addresses:* liumehuan@163.com (M. Liu), shushi@xtu.edu.cn (S. Shu), yuan_guangwei@iapcm.ac.cn (G. Yuan), yuexq@xtu.edu.cn (X. Yue)

1 Introduction

The heat conduction equations are derived from a wide variety of applications, see, e.g., inertial confinement fusion and astrophysics [3, 8]. Finite volume discretizations are frequently used for the consideration of local conservations [6, 9, 10, 20, 21]. The positivity-preserving property plays a crucial role in the design of discretization methods. Many non-physical phenomena including negative discrete concentrations, densities and temperatures (in Kelvin) would emerge into the simulation in case the used discretization does not ensure the positivity-preserving property.

The work of Nordbotten, Aavatsmark and Eigestad [26] showed that there are no linear schemes with nine-point stencil on structured quadrilateral meshes which both preserve the positivity of the continuous solution and local conservation with second-order accuracy. The nonlinear approximation might be a price to pay for these two requirements. Over the past decades, a number of nonlinear positivity-preserving finite volume (PPFV) schemes have been originally proposed for two-dimensional (2-D) diffusion problems. To our knowledge, Bertolazzi and Manzini [2] first developed a nonlinear combination of two one-sided fluxes and then constructed a class of second-order extremum-preserving schemes only on simplex meshes. Le Potier [19] proposed a nonlinear PPFV scheme well-suited for parabolic problems on unstructured triangular meshes, however, with some restrictions on the time step-size. This scheme was generalized to general triangular and regular-shaped polygonal meshes in [22], where the cell-centers should be located relevantly to the diffusion coefficient. Physical quantity remappings are needed when the generalized scheme is applied to the arbitrary Lagrangian-Eulerian radiation hydrodynamics simulation, which would bring about additional numerical errors. To remedy this, Yuan and Sheng [36] presented an adaptive stencil and a convex decomposition for the co-normal vectors and then established a nonlinear PPFV scheme on arbitrary polygonal meshes with star-shaped cells by incorporating a certain nonlinear combination. Further ameliorations and applications have since been appeared, such as the auxiliary unknowns were located at edge midpoints as opposed to vertices in [29], the concept of identical collocation was introduced to avoid using the nonnegative assumption on auxiliary unknowns in [30] and the application to nonequilibrium radiation diffusion equations was examined in [31] and to advection diffusion cases in [33]. New schemes preserving the positivity property are constructed by taking both the geometric deformation of the underlying mesh and the change of physical quantity into account. Utilizing some characteristics of the quadrilateral mesh, a positivity-preserving scheme with relatively fixed stencil was obtained in [37]. Moreover, a nonlinear PPFV scheme with fixed stencil was framed in [13] by exploiting certain nonlinear convex combinations, and a vertex-centered nonlinear PPFV scheme with fixed stencil was generalized recently in [32] for three-temperature radiation diffusion equations on general polygonal meshes.

There has also been much research devoted to building the nonlinear PPFV schemes for three-dimensional (3-D) problems. Danilov and Vassilevski [5] generalized the 2-D