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Acoustic Scattering Problems with Convolution Quadrature and the Method of Fundamental Solutions

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Abstract. Time-domain acoustic scattering problems in two dimensions are studied. The numerical scheme relies on the use of the Convolution Quadrature (CQ) method to reduce the time-domain problem to the solution of frequency-domain Helmholtz equations with complex wavenumbers. These equations are solved with the method of fundamental solutions (MFS), which approximates the solution by a linear combination of fundamental solutions defined at source points inside (outside) the scatterer for exterior (interior) problems. Numerical results show that the coupling of both methods works efficiently and accurately for multistep and multistage based CQ.

AMS subject classifications: 65N35, 65N80, 78M25

Key words: Acoustic wave scattering, convolution quadrature, method of fundamental solutions.

1 Introduction

For several years now, since the seminal work of Ch. Lubich [20,21], convolution quadrature (CQ) has attracted considerable attention as a method for simulating wave propagation in time domain, based on relationships obtained in the Laplace domain. This approach was successful in the wave propagation context, because it allows the use of frequency-domain Green's functions instead of their more complicated time-domain counterparts (see Eq. (1.2)).

This makes possible the use of integral equation methods to solve the arising frequency-domain problems. Integral equation methods are also an option in time-domain, but they require evaluation of complicated integrals in two dimensions, and dealing with distributional expressions in three dimensions [17]. Instead of resorting to

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integral equations, we follow a different and simpler approach. This is the method of fundamental solutions (MFS) [5,13], which assumes that the solution of the Helmholtz equation can be represented by a linear combination of fundamental solutions with sources located at the interior of the scatterer. The advantage of using MFS is that meshing the volume is not required for solving the problem, in contrast to Perfectly-Matched-Layer (PML) approaches and Absorbing Boundary Conditions (ABCs). The MFS solution also satisfies the exact radiation conditions, a property inherited from the fundamental solution. It can be used for general scatterers, not necessarily convex. Compared to integral equation methods, MFS does not require singular integrals to be computed. When combined with the CQ scheme it makes possible to sparsify of the resulting matrix, thanks to the exponential decay of fundamental solutions for modified Helmholtz equations [9]. It is also possible to use techniques developed for Boundary Element methods, which allow the use of directional $\mathcal{H}$-matrices for each of the Helmholtz problems [8]. We acknowledge that a disadvantage of the MFS is that the resulting systems are ill-conditioned usually. Also, the appropriate placement of source points is a challenge, in particular for 3D problems.

Although a combination of MFS with Laplace transform techniques and the modified Helmholtz equation has been mentioned before [9, 19], as far as we know, there are no results for a successful implementation of MFS in combination with multistep and multistage methods in time domain, on which CQ is based.

The problem that we are interested in solving is the exterior (interior) acoustic scattering problem in the time-domain. Let $\Omega\subset\mathbb{R}^d$, $d=2,3$, be the bounded region of space occupied by the scatterer. By rescaling we can achieve that the wave speed in the homogeneous exterior domain $\mathbb{R}^d\backslash\overline{\Omega}$ is given by $c=1$. The equations for the scattered field u excited by an incident wave u^{inc} are as follows [23, Section 1.5]

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t)-\Delta u(\boldsymbol{x},t)=0, & \boldsymbol{x}\in\mathbb{R}^d\backslash\overline{\Omega}\ \ (\text{or}\ \ \boldsymbol{x}\in\Omega), \quad t\geq 0,\\ u(\boldsymbol{x},t)=-u^{\mathrm{inc}}(\boldsymbol{x},t), & \boldsymbol{x}\in\Gamma:=\partial\Omega, \qquad\qquad t\geq 0,\\ u(\boldsymbol{x},0)\ =\ \dfrac{\partial u}{\partial t}(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\mathbb{R}^d\backslash\overline{\Omega}\ \ (\text{or}\ \ \boldsymbol{x}\in\Omega). \end{cases} \tag{1.1}$$

This is a wave equation with Dirichlet boundary conditions and zero initial conditions. At initial time $t=0$ it is assumed that the incident field has not yet reached the scatterer, which is reflected by our initial conditions.

The fundamental solutions for the wave equation in two and three dimensions centered at a given point $\boldsymbol{y}\in\Omega$ read

$$\mathcal{G}(\boldsymbol{x}-\boldsymbol{y},t)=\begin{cases} \dfrac{H(t-|\boldsymbol{x}-\boldsymbol{y}|)}{2\pi\sqrt{t^2-|\boldsymbol{x}-\boldsymbol{y}|^2}}, & d=2,\\ \dfrac{\delta(t-|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}, & d=3, \end{cases} \tag{1.2}$$