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On the Five-Moment Maximum Entropy System of One-Dimensional Boltzmann Equation

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Abstract. The maximum entropy moment system extends the Euler equation to non-equilibrium gas flows by considering higher order moments such as the heat flux. This paper presents a systematic study of the maximum entropy moment system of Boltzmann equation. We consider a hypothetical one-dimensional gas and study a five-moment model. A numerical algorithm for solving the optimization problem is developed to produce the maximum entropy distribution function from known moments, and the asymptotic behaviour of the system around the singular region known as the Junk's line, as well as that near the boundary of the realizability domain is analyzed. Furthermore, we study the properties of the system numerically, including the behaviour of the system around the Maxwellian and within the interior of the realizability domain, and properties of its characteristic fields. Our studies show the higher order entropy-based moment models to differ significantly from the Euler equations. Much of this difference comes from the singularity near the Junk's line, which would be removed if a truncation of the velocity domain is employed.

AMS subject classifications: 76P05, 82B40, 82D05

Key words: Maximum entropy method, five-moment system, characteristic structure, Boltzmann equation.

1 Introduction

The study of rarefied gas flows is an important topic in the field of computational fluid dynamics due to its many applications such as space shuttle reentry and design of equipment for semiconductor chips. An important concept in the field of rarefied gas dynamics is the Knudsen number, Kn, which is defined as the ratio between the particle mean

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free path and the characteristic length scale, and commonly used to described the denseness of particles. With Knudsen number at below 10^{-2} within the fluid dynamic regime, the fluid behavior could be well described by hydrodynamic equations like the Euler or Navier-Stokes equations. However, as the Knudsen number becomes larger and enters the transition regime at around $\mathcal{O}(0.1) \sim \mathcal{O}(10)$, traditional hydrodynamic models are known to fail to give accurate characterizations of the fluid flow. On the other hand, the Boltzmann equation can accurately describe the nonequilibrium effect in the microscopic particle velocity distribution of rarefied gases. However, due to its high dimensionality, direct computation with the Boltzmann equation can bring in a lot of computational cost.

The moment method derives evolution equations of macroscopic quantities of the gas by projecting the distribution function onto a finite dimensional polynomial subspace in velocity. It provides a natural framework for approximation of the Boltzmann equation and extending the validity of the hydrodynamic equations into the transition regime. Moment methods provide us a practical way to reduce equations, such as Boltzmann equation [10, 11, 20, 21, 35], radiative transfer equation [19, 22, 23, 40], shallow flow models [30] etc. Traditional moment methods include the Grad expansion method [20,21] and the Chapman-Enskog expansion method [13, 18]. However, both methods suffer from problems which limit their applications. High order Chapman-Enskog expansion, such as the Burnett and super-Burnett equations, are generally unstable [9]; and Grad's moment equations suffer from loss of hyperbolicity [12] and lead to unphysical sub-shocks when the Mach number is large [50].

Another type of moment method is the maximum entropy moment method [15, 25, 35, 41]. It is based on the idea that the approximate form of the distribution function is the one which, among all functions which satisfy the given moments, maximizes the physical entropy. Levermore [35] derived a whole hierarchy of relaxation systems from the Boltzmann equation based on the above maximum entropy principle. This system has the Euler equation as first member, and satisfy remarkable properties such as global hyperbolicity and entropy dissipation. Also, its velocity distribution is non-negative. Since the study of the maximum entropy model by Levermore [35], its idea has attracted widespread attention. Analysis of the maximum entropy model include its existence of solutions [5, 26, 42, 48, 52], its boundary conditions [33], and characterization of its moment spaces [32]. In a series of work [1–4], Abramov developed a practical framework for computing the maximum entropy moment problem. Based on this work, Schaerer and Torrilhon et al. considered simulation of the maximum entropy model [45, 47]. Extensions to the maximum entropy moment model studied by Levermore include application of the maximum entropy principle to derive moment models for polyatomic gases [6, 43, 44], and application to other fields like the radiative transfer [17]. There have also been numerous work to approximate or modify the maximum entropy model [16, 31, 37, 38, 46, 51]. However, for maximum entropy systems containing higher than second-order moments, the flux functions are not available in closed form and numerically evaluating the flux functions involve solving expensive optimization problem and computing numerical quadratures. Moreover, as shown by Junk [26, 28] and analyzed