

# A Modified Adaptive Improved Mapped WENO Method

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**Abstract.** We propose a new family of mapped WENO schemes by using several adaptive control functions and a smoothing approximation of the signum function. The proposed schemes admit an extensive permitted range of the parameters in the mapping functions. Consequently, they have the capacity to achieve optimal convergence rates, even near critical points. Particularly, the new schemes with fine-tuned parameters illustrates a significant advantage when solving problems with discontinuities. It produces numerical solutions with high resolution without generating spurious oscillations, especially for long output times.

**AMS subject classifications:** 65M06, 65M12

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## 1 Introduction

In recent decades, the essentially non-oscillatory (ENO) schemes [1–4] and weighted ENO (WENO) schemes [5–9] have been developed quite successfully to solve the hyperbolic conservation laws in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{\alpha=1}^d \frac{\partial \mathbf{f}_{\alpha}(\mathbf{u})}{\partial x_{\alpha}} = 0, \quad x_{\alpha} \in \mathbb{R}, \quad t > 0, \quad (1.1)$$

where  $\mathbf{u} = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$  are the conserved variables and  $\mathbf{f}_{\alpha}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $\alpha = 1, 2, \dots, d$  are the Cartesian components of flux. Within the general framework (referred as WENO-JS below) of smoothness indicators and non-linear weights proposed by Jiang and Shu

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[8], many successful works have improved the accuracy [10–14] and efficiency [15–19] of WENO schemes.

Henrick et al. [10] found that the fifth-order WENO-JS scheme is only third-order accurate at critical points of order  $n_{cp}=1$  in the smooth regions, where  $n_{cp}$  denotes the order of the critical point; e.g.,  $n_{cp}=1$  corresponds to  $f'=0, f'' \neq 0$  and  $n_{cp}=2$  corresponds to  $f'=0, f''=0, f''' \neq 0$ , etc. They derived necessary and sufficient conditions [10] on the weights for optimality of the order. Then, by introducing a mapping function to the original weights of the WENO-JS scheme, they developed the WENO-M method [10] to achieve the optimal order of accuracy in smooth regions even with critical points. Feng et al. pointed out that [12], when the mapping function of the WENO-M scheme is used for solving the problems with discontinuities, it may amplify the effect from the non-smooth stencils, thereby causing a potential loss of accuracy near discontinuities. In order to address this issue, they devised the WENO-PM $k$  scheme [12] by proposing a piecewise polynomial mapping function with two additional requirements, that is,  $g'(0)=0$  and  $g'(1)=0$  ( $g(x)$  denotes the mapping function), to the original criteria in [10]. Also, these two additional requirements were considered to be very important to decrease the effect from the non-smooth stencils [20], and they were used in the construction of the WENO-RM( $mn0$ ) scheme [20]. Similarly, requirements  $g'(0)=1$  and  $g'(1)=1$  were employed when the WENO-PPM $n$ ( $n=4,5,6$ ) [21] schemes were constructed. Although the mapping function in the WENO-PM $k$  scheme decreases the effect from the non-smooth stencils, it is not smooth enough [20] because it is only piecewise continuous. Furthermore, the WENO-PM $k$  scheme may generate the oscillations near discontinuities [20]. Recently, Feng et al. [11] proposed a new family of mapping functions, that has two parameters  $k$  and  $A$ , to improve the WENO-M method. They called the corresponding improved mapped WENO scheme WENO-IM( $k, A$ ). The WENO-IM( $k, A$ ) scheme with proper parameters can achieve optimal order of accuracy near critical points for any  $(2r-1)$ th-order WENO schemes. Moreover, the recommended version of the WENO-IM( $k, A$ ) scheme, that is, WENO-IM(2,0.1), provides better numerical solutions [11] with less dissipation and higher resolution for the fifth-order WENO method than the WENO-JS, WENO-M and WENO-Z [15] schemes. However, it is required that the parameter  $k$  has to be an even integer [11] in the WENO-IM( $k, A$ ) method, which may prevent one from finding the best version from the family of the WENO-IM( $k, A$ ) schemes. Besides, it is easy to verify that the mapping function of the WENO-IM( $k, A$ ) method could not satisfy the requirements, namely  $g'(0)=0$  or  $g'(0)=1$ , to decrease the effect from the non-smooth stencils, which may lead to the numerical solution with non-physical oscillations near discontinuities [20]. It was demonstrated by numerical experiments [20] that the seventh- and ninth- order WENO-IM(2,0.1) schemes generate the oscillations obviously when solving the linear advection problem with discontinuities proposed by Jiang and Shu [8] for a long output time. Actually, we found that the fifth-order WENO-IM(2,0.1) also generates the oscillations when solving the linear advection problem with discontinuities by taking a long output time and a *bigger* grid number, and we show the numerical results in subsection 4.2 of this paper. Therefore, the goal of this study is to