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High Order Finite Difference Hermite WENO Fixed-Point Fast Sweeping Method for Static Hamilton-Jacobi Equations

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Abstract. In this paper, we combine the nonlinear HWENO reconstruction in [43] and the fixed-point iteration with Gauss-Seidel fast sweeping strategy, to solve the static Hamilton-Jacobi equations in a novel HWENO framework recently developed in [22]. The proposed HWENO frameworks enjoys several advantages. First, compared with the traditional HWENO framework, the proposed methods do not need to introduce additional auxiliary equations to update the derivatives of the unknown function ϕ. They are now computed from the current value of ϕ and the previous spatial derivatives of ϕ. This approach saves the computational storage and CPU time, which greatly improves the computational efficiency of the traditional HWENO scheme. In addition, compared with the traditional WENO method, reconstruction stencil of the HWENO methods becomes more compact, their boundary treatment is simpler, and the numerical errors are smaller on the same mesh. Second, the fixed-point fast sweeping method is used to update the numerical approximation. It is an explicit method and does not involve the inverse operation of nonlinear Hamiltonian, therefore any Hamilton-Jacobi equations with complex Hamiltonian can be solved easily. It also resolves some known issues, including that the iterative number is very sensitive to the parameter ε used in the nonlinear weights, as observed in previous studies. Finally, to further reduce the computational cost, a hybrid strategy is also presented. Extensive numerical experiments are performed on two-dimensional problems, which demonstrate the good performance of the proposed fixed-point fast sweeping HWENO methods.

AMS subject classifications: 65M60, 35L65

Key words: Hermite method, Weighted Essentially Non-Oscillatory (WENO) method, fixed-point iteration, Hamilton-Jacobi equation, hybrid strategy.

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1 Introduction

The static Hamilton-Jacobi (HJ) equations often appear in many application fields, for instance in optimal control, computer vision, differential game, geometric optics, image processing and so on [7,31]. The general static HJ equations have the form

$$\begin{cases} H(\nabla\phi,\mathbf{x})=0, & \mathbf{x}\in\Omega\backslash\Gamma, \\ \phi(\mathbf{x})=g(\mathbf{x}), & \mathbf{x}\in\Gamma\subset\Omega, \end{cases} \tag{1.1}$$

where Ω is the computational domain in $\mathbb{R}^d$, $\phi(\mathbf{x})$ is the unknown function in Ω, the Hamiltonian H is a nonlinear Lipschitz continuous function depending on $\nabla\phi$ and $\mathbf{x}$, and the boundary condition is given by $\phi(\mathbf{x})=g(\mathbf{x})$ on the subset $\Gamma\subset\Omega$. Eikonal equation is a prototype example of the static HJ equations, taking the form

$$\begin{cases} |\nabla\phi|=f(\mathbf{x}), & \mathbf{x}\in\Omega\backslash\Gamma, \\ \phi(\mathbf{x})=g(\mathbf{x}), & \mathbf{x}\in\Gamma\subset\Omega, \end{cases} \tag{1.2}$$

where $f(\mathbf{x})>0$. It can be derived from Maxwell's electromagnetic equations and provides a link between physical optics and geometric optics.

In general, the global C^1 solution does not exist for the time-dependent nonlinear HJ equations, even if the initial condition is sufficiently smooth. Singularities in the form of discontinuities would appear in the derivatives of the unknown function, hence it is necessary to define a "weak solution" for the HJ equations. The viscosity solutions of the HJ equations were first introduced by Crandall and Lions in [3].

There are mainly two types of numerical methods to solve the static HJ equations. The first one is to solve the following equation

$$\phi_t+H(\nabla\phi)=f(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d,$$

with pseudo-time iteration. The equation is evolved in time [25] until the numerical solution converges to a steady state. However, such method requires a very large number of iterations to obtain the convergence of the solution in the entire domain, with the main reason being the finite speed of propagation, and the restrictive CFL time step requirement for stability. The second popular algorithm is to treat the problem as a stationary boundary value problem, so that the fast marching method (FMM) [5,24,28] or the fast sweeping method (FSM) [9,17,27,37] can be applied. FSM can be constructed to be high order accurate, and becomes a class of popular and effective methods for solving static HJ equations nowadays. The FSM was first proposed in [1] by Boué and Dupuis when solving a deterministic control problem with quadratic running cost using Markov chain approximation. Later, Zhao [37] studied the FSM for the Eikonal equation, and demonstrate the efficiency and effectiveness of the method. Since then, many high order FSM have been proposed to solving static HJ equations. In the framework