Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2021-0062

Vol. **31**, No. 1, pp. 27-59
January 2022

A Coupled FEM-BEM Approach for the Solution of the Free-Boundary Axi-Symmetric Plasma Equilibrium Problem

M. Bonotto[1,2,*], D. Abate[1], P. Bettini[1,3] and F. Villone[4]

[1] *Consorzio RFX (CNR, ENEA, INFN, Università di Padova, Acciaierie Venete SpA), C.so Stati Uniti 4, 35127 Padova, Italy.*
[2] *INFN-LNL, Legnaro, Italy.*
[3] *Dipartimento di Ingegneria Industriale (DII), Università di Padova, Italy.*
[4] *Consorzio CREATE, DIETI, Università di Napoli Federico II, Italy.*

Received 29 March 2021; Accepted (in revised version) 27 Septemger 2021

Abstract. In this paper we present a coupled Finite Element Method – Boundary Element Method (FEM-BEM) approach for the solution of the free-boundary axi-symmetric plasma equilibrium problem. The proposed method, obtained from an improvement of the Hagenow-Lackner coupling method, allows to efficiently model the equilibrium problem in unbounded domains by discretizing only the plasma region; the external conductors can be modelled either as 2D or 3D models, according to the problem of interest. The paper explores different iterative methods for the solution of the nonlinear Grad-Shafranov equation, such as Picard, Newton-Raphson and Newton-Krylov, in order to provide a robust and reliable tool, able to handle large-scale problems (e.g. high resolution equilibria). This method has been implemented in the FRIDA code (FRee-boundary Integro-Differential Axisimmetric – `https://github.com/matteobonotto/FRIDA`), together with a suitable Adaptive Integration Technique (AIT) for the computation of the source term. FRIDA has been successfully tested and validated against experimental data from RFX-mod device, and numerical equilibria of an ITER-like device.

PACS: 52.65.Kj, 52.55.Fa, 52.55.Hc, 41.20.Gz
Key words: FEM-BEM, MHD, plasma equilibrium, Grad-Shafranov equation.

1 Introduction

In magnetic confinement fusion (MCF) research, the successful design of Tokamak devices, the set up of plasma operations, the prediction of performance scenarios and the

*Corresponding author. *Email addresses:* `matteo.bonotto@igi.cnr.it` (M. Bonotto), `domenico.abate@igi.cnr.it` (D. Abate), `paolo.bettini@unipd.it` (P. Bettini), `fabio.villone@unina.it` (F. Villone)

design of feedback control systems are all activities which rely, to an extent, on equilibrium codes [1, 2]. High accuracy MHD stability analysis also relies on given equilibrium reconstructions, which are usually performed in fixed-boundary conditions. One of the basic problems of equilibrium reconstructions is the more general quasi-static *free-boundary* plasma equilibrium problem, consisting in the numerical solution of the Grad-Shafranov equation [3, 4] with a plasma separatrix, the *plasma-vacuum interface*, which is not known a priori.

The solution of the free-boundary axi-symmetric equilibrium problem requires to deal with a two-dimensional, elliptic partial differential equation (PDE), defining the axi-symmetric plasma equilibrium, given the prescribed set of external currents, the plasma current density profile and the total plasma current value. The computational challenges in solving such magneto-quasistatic problem are related to the fact that (i) the computational domain is unbounded, (ii) the plasma current density profiles is a non-linear function of the poloidal magnetic flux and (iii) the plasma separatrix is unknown.

Despite the well-assessed theory concerning the axisymmetric quasi-static plasma equilibrium problem [5, 6], its numerical solution is still a hot topic. Many equilibrium codes are based on Finite-Difference (FD) or Finite-Element-Method (FEM) for the spatial discretization of the PDE, a fixed-point (Picard) iteration scheme to solve the non-linearities, and a coupling method, based on the analytical Green's functions [7], to reduce the unbounded domain to a bounded region. Among these codes, some examples are FBT [8], MAXFEA [9], and TES [10]. Other computational tools are based on a Newton-Raphson like iterative scheme, and a more sophisticated analytic uncoupling on a semi-circular domain, which was introduced by Albanese, Blum and de Barbieri [11]. This approach is implemented, for examples, in the codes PROTEUS [12], CREATE-L [13], CREATE-NL+ [14], CEDRES++ [15] and FEEQS.M [5]. All these codes can be divided in two groups, depending on how the plasma equilibrium problem is treated. On one hand, we find all the tools based on robust and reliable Newton-Raphson like schemes used to solve the non-linear equilibrium problem in a rigorous way (i.e PROTEUS, CREATE-L, CREATE-NL+, CEDRES++, and FEEQS.M). On the other hand, a group of codes (i.e. FBT, MAXFEA, and TES) rely on the idea that the free-boundary equilibrium problem is characterized by an *intrinsic axisymmetric instability which is encountered in all equilibrium calculations with a free-boundary condition*. This misconception is explicitly expressed in [10], and more indirectly in [16], and it is pretty common in the MCF community, with the consequence of believing that it is impossible to solve the free-boundary plasma equilibrium problem without handling this *intrinsic axisymmetric instability* by means of naive numerical approaches (e.g. with the insertion of a feedback loop). However, such misconceptions come from a misinterpretation of the convergence limitations of the Picard scheme, much less robust and reliable than Newton-Raphson-like scheme, as will be clarified in sections 5 and 6.

In the sake of a rigorous mathematical analysis of the problem [5, 6], in this paper we present a coupled Finite Element Method – Boundary Element Method (FEM-BEM) approach obtained improving the well-known Hagenow-Lackner (HL) coupling method