Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2021-0127

Vol. **31**, No. 1, pp. 94-130
January 2022

A Well-Balanced Positivity-Preserving Quasi-Lagrange Moving Mesh DG Method for the Shallow Water Equations

Min Zhang[1], Weizhang Huang[2] and Jianxian Qiu[3,*]

[1] *School of Mathematical Sciences, Peking University, Beijing, 100871, China.*
[2] *Department of Mathematics, University of Kansas, Lawrence, Kansas 66045, USA.*
[3] *School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling and High-Performance Scientific Computing, Xiamen University, Xiamen, Fujian 361005, P.R. China.*

Received 13 June 2021; Accepted (in revised version) 27 September 2021

Abstract. A high-order, well-balanced, positivity-preserving quasi-Lagrange moving mesh DG method is presented for the shallow water equations with non-flat bottom topography. The well-balance property is crucial to the ability of a scheme to simulate perturbation waves over the lake-at-rest steady state such as waves on a lake or tsunami waves in the deep ocean. The method combines a quasi-Lagrange moving mesh DG method, a hydrostatic reconstruction technique, and a change of unknown variables. The strategies in the use of slope limiting, positivity-preservation limiting, and change of variables to ensure the well-balance and positivity-preserving properties are discussed. Compared to rezoning-type methods, the current method treats mesh movement continuously in time and has the advantages that it does not need to interpolate flow variables from the old mesh to the new one and places no constraint for the choice of a update scheme for the bottom topography on the new mesh. A selection of one- and two-dimensional examples are presented to demonstrate the well-balance property, positivity preservation, and high-order accuracy of the method and its ability to adapt the mesh according to features in the flow and bottom topography.

AMS subject classifications: 65M50, 65M60, 76B15, 35Q35

Key words: Well-balance, positivity-preserving, high-order accuracy, quasi-Lagrange moving mesh, DG method, shallow water equations.

---

*Corresponding author. *Email addresses:* `minzhang@math.pku.edu.cn` (M. Zhang), `whuang@ku.edu` (W. Huang), `jxqiu@xmu.edu.cn` (J. Qiu)

1 Introduction

The shallow water equations (SWEs) model the water flow over a surface such as hydraulic jumps/shocks and open-channel flows in the ocean/hydraulic engineering. They can be derived by integrating the Navier-Stokes equations in depth under the hydrostatic assumption when the depth of the flow is small compared to its horizontal dimensions. The two-dimensional SWEs can be cast in conservative form as

$$V_t+\nabla\cdot\mathcal{F}(V)=\mathcal{S}(h,B), \tag{1.1}$$

where h is the depth of water, $V=(h,m,w)^T$ denote the conservative variables, $(m,w)=(hu,hv)$ are the discharges, (u,v) are the velocities, $B=B(x,y)$ is the bottom topography assumed to be a given time-independent function, g is the gravitation acceleration, and the flux $\mathcal{F}(V)$ and the source $\mathcal{S}(h,B)$ are given by

$$\mathcal{F}(V)=\begin{bmatrix} m & w \\ \frac{m^2}{h}+\frac{1}{2}gh^2 & \frac{mw}{h} \\ \frac{mw}{h} & \frac{w^2}{h}+\frac{1}{2}gh^2 \end{bmatrix}, \quad \mathcal{S}(h,B)=\begin{bmatrix} 0 \\ -hgB_x \\ -hgB_y \end{bmatrix}. \tag{1.2}$$

An illustration of h, B, and the free water surface level $\eta=h+B$ is given in Fig. 1.

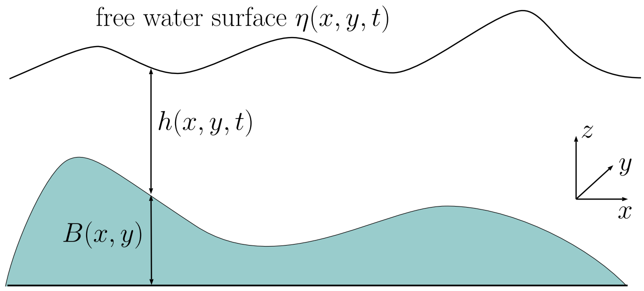

Figure 1: An illustration of the water depth h, the bottom topography B, and the free water surface level $\eta=h+B$.

We are interested in the preservation of the "lake-at-rest" steady state solution

$$m=hu=0, \quad w=hv=0, \quad \eta=h+B=C, \tag{1.3}$$

where C is a constant. Many physical phenomena can be described as small perturbations of this steady-state solution, including waves on a lake or tsunami waves in the deep ocean. They are difficult, if not impossible, to capture by a numerical method that does not preserve (1.3), on an unrefined mesh. Thus, for the numerical simulation of perturbation waves over the lake-at-rest steady state, it is important to develop schemes that preserve (1.3). These schemes are said in literature to be well-balanced or have the well-balance property or the C-property. Bermudez and Vazquez [3] first introduced