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Fractional Buffer Layers: Absorbing Boundary Conditions for Wave Propagation

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Abstract. We develop fractional buffer layers (FBLs) to absorb propagating waves without reflection in bounded domains. Our formulation is based on variable-order spatial fractional derivatives. We select a proper variable-order function so that dissipation is induced to absorb the coming waves in the buffer layers attached to the domain. In particular, we first design proper FBLs for the one-dimensional one-way and two-way wave propagation. Then, we extend our formulation to two-dimensional problems, where we introduce a consistent variable-order fractional wave equation. In each case, we obtain the fully discretized equations by employing a spectral collocation method in space and Crank-Nicolson or Adams-Bashforth method in time. We compare our results with a finely tuned perfectly matched layer (PML) method and show that the proposed FBL is able to suppress reflected waves including corner reflections in a two-dimensional rectangular domain. We also demonstrate that our formulation is more robust and uses less number of equations.

AMS subject classifications: 35L05, 26A33, 76M22

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1 Introduction

Waves are omnipresent in nature in diverse physical and biological phenomena. They are governed by first-order, second-order or higher-order partial differential equations

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(PDEs) that lead to one-way or two-way waves, giving rise to oscillating solutions that propagate through spatio-temporal domains while conserving energy in lossless media. Examples include the scalar wave equations for pressure waves in gases, Maxwell's equations in electromagnetism, Schrödinger's equation in quantum mechanics, and elastic vibrations. In solving numerically the governing equations of these examples on a finite domain, it is necessary to truncate the computational domain in a certain way that does not introduce significant artifacts into the computation. Therefore, an efficient model for propagation of waves from the interior of a finite domain and absorption through its boundaries is desirable in numerous physical problems [3]. Several approaches have been studied in the literature. Some works have focused on finding analytical boundary conditions for the differential equations and then discretizing the analytical conditions [39, and references therein]. Some other works are dedicated to constructing absorbing boundary conditions via directly working with approximations to the wave equation [11, 12, and references therein]. A well-known approach in the literature is the perfectly matched layer (PML) method, which in theory absorbs strongly the outgoing waves from the interior of a computational domain without reflecting them back into the interior. This method was first introduced by J.P. Bérenger [4], and thereafter a number of works based on PML and corresponding numerical schemes were developed; see for example [9, 13, and references therein]. A comprehensive introduction to the PML for wave equations can be found in [15]. Other modifications to the PML method and the numerical methods can be found in [5, 10, 14, 22, 25, 30, 34, 37, 38, 40].

We develop a new absorbing layer, namely a *fractional buffer layer* (FBL), by exploiting the flexibility and expressivity of variable-order fractional differential operators. It is widely recognized that fractional calculus has important applications in various scientific fields. In particular, the fractional derivatives that extend the notion of their integer-order counterparts have been shown to provide a powerful mathematical tool that can be used to describe many physical problems, especially anomalous transport [2, 23, 29, 36, 42]. In a more general setting, variable-order fractional derivatives further extend the fixed fractional order to a spatio-temporal variable function. They can accurately describe the multi-scale behavior of systems with temporally/spatially varying properties [6, 7, 14, 30–32].

One of the interesting applications of the variable-order fractional differential operators is in modeling wave propagation in finite domains. In [46], a variable-order time fractional differential operator is employed to control wave reflections in truncated computational domains by switching from a wave- to a diffusion-dominated equation at the boundaries. In that approach, a priori knowledge of the time that waves reach the boundary is needed to effectively design the variable-order function. In the current paper, we consider the application of variable-order space fractional differential operators, where the variable-order function is solely a function of space variables. We formulate our method by extending the space domain and attaching a FBL to its boundaries. Then, we define a proper variable-order function such that we recover the original equation inside the interior domain of interest and introduce dissipation in the buffer layer to absorb