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An LDG Method for Stochastic Cahn-Hilliard Type Equation Driven by General Multiplicative Noise Involving Second-Order Derivative

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Abstract. In this paper, we propose a local discontinuous Galerkin (LDG) method for the multi-dimensional stochastic Cahn-Hilliard type equation in a general form, which involves second-order derivative Δu in the multiplicative noise. The stability of our scheme is proved for arbitrary polygonal domain with triangular meshes. We get the sub-optimal error estimate $\mathcal{O}(h^k)$ if the Cartesian meshes with Q^k elements are used. Numerical examples are given to display the performance of the LDG method.

AMS subject classifications: 65C30, 60H35

Key words: Local discontinuous Galerkin method, stochastic Cahn-Hilliard type equations, multiplicative noise, stability analysis, error estimates.

1 Introduction

The Cahn-Hilliard equation was originally introduced in [2, 3] to describe complicated phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably. In this paper, we consider the following general stochastic Cahn-Hilliard type equation with homogeneous Neumann boundary condition, whose multiplicative noise non-linearly involves

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the second-order derivative Δu in a general form:

$$\begin{cases} du = \Delta w dt + g(\cdot,u,\nabla u,\Delta u) dW_t, & \text{in } D\times(0,T], \\ w = -a(\cdot,u,\nabla u,\Delta u), & \text{in } D\times(0,T], \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial w}{\partial \vec{n}} = 0, & \text{on } \partial D\times(0,T], \\ u(0,\cdot) = u_0(\cdot), & \text{in } D, \end{cases} \tag{1.1}$$

where $D\subseteq\mathbb{R}^d$ is a polygonal domain, $\vec{n}$ stands for the unit outward normal to ∂D, and the terminal time $T>0$ is a fixed number. Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{0\le t\le T},\mathbb{P})$ be a complete filtered probability space on which a real-valued Wiener process $W=\{W_t; t\ge 0\}$ is defined. The stochastic initial value $u_0: \Omega\times D\longrightarrow\mathbb{R}$ is given, and for each fixed $x\in D$, $u_0(\cdot,x)$ is independent with W. The real-valued functions $a,g: \Omega\times[0,T]\times D\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\longrightarrow\mathbb{R}$ are given. For each $(x,u,\vec{v},q)\in D\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$, the stochastic processes $a(\cdot,\cdot,x,u,\vec{v},q)$ and $g(\cdot,\cdot,x,u,\vec{v},q)$ are adapted with respect to $\{\mathscr{F}_t\}_{t\ge 0}$. We make also the following hypotheses:

(H1) These exist two constants $\alpha>0$ and $L_a>0$ such that the function a satisfies the monotone condition

$$a\Big(\omega,t,x,0,\vec{0},q\Big)q\ge\alpha|q|^2,\quad \left[a(\omega,t,x,u,\vec{v},q)-a\left(\omega,t,x,u,\vec{v},q'\right)\right]\left(q-q'\right)\ge\alpha\left|q-q'\right|^2,$$

and satisfies the Lipschitz continuous condition

$$\left|a(\omega,t,x,u,\vec{v},q)-a\left(\omega,t,x,u',\vec{v}',q'\right)\right|\le L_a\left(\left|u-u'\right|+\left|\vec{v}-\vec{v}'\right|+\left|q-q'\right|\right),$$

for each $(\omega,t,x,u,u',\vec{v},\vec{v}',q,q')\in\Omega\times[0,T]\times D\times\mathbb{R}^2\times\mathbb{R}^{2d}\times\mathbb{R}^2$.

(H2) There exists a constant $L_g>0$ such that the function g satisfies the Lipschitz continuous condition

$$\left|g(\omega,t,x,u,\vec{v},q)-g\left(\omega,t,x,u',\vec{v}',q'\right)\right|\le L_g\left(\left|u-u'\right|+\left|\vec{v}-\vec{v}'\right|+\left|q-q'\right|\right),$$

with $\left|g\left(\omega,t,x,0,\vec{0},0\right)\right|\le L_g$, for each $(\omega,t,x,u,u',\vec{v},\vec{v}',q,q')\in\Omega\times[0,T]\times D\times\mathbb{R}^{2\times 2d\times 2}$.

(H3) For the above constants α and L_g, the uniformly stochastic elliptic condition holds as follows

$$2\alpha-L_g^2>0.$$

Concerning the theoretical research for stochastic Cahn-Hilliard equation (1.1), Krylov and Rozovskii [17] first investigated the general solvability theory under the framework of Hilbert spaces by applying abstract results. After that, The existence and uniqueness of a function-valued process solution to stochastic Cahn-Hilliard equation driven by a nonlinear multiplicative noise was studied by Cardon-Weber in [4]. Recently, Wang and