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VPVnet: A Velocity-Pressure-Vorticity Neural Network Method for the Stokes' Equations under Reduced Regularity

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Abstract. We present VPVnet, a deep neural network method for the Stokes' equations under reduced regularity. Different with recently proposed deep learning methods [40,51] which are based on the original form of PDEs, VPVnet uses the least square functional of the first-order velocity-pressure-vorticity (VPV) formulation ([30]) as loss functions. As such, only first-order derivative is required in the loss functions, hence the method is applicable to a much larger class of problems, e.g. problems with non-smooth solutions. Despite that several methods have been proposed recently to reduce the regularity requirement by transforming the original problem into a corresponding variational form, while for the Stokes' equations, the choice of approximating spaces for the velocity and the pressure has to satisfy the LBB condition additionally. Here by making use of the VPV formulation, lower regularity requirement is achieved with no need for considering the LBB condition. Convergence and error estimates have been established for the proposed method. It is worth emphasizing that the VPVnet method is divergence-free and pressure-robust, while classical inf-sup stable mixed finite elements for the Stokes' equations are not pressure-robust. Various numerical experiments including 2D and 3D lid-driven cavity test cases are conducted to demonstrate its efficiency and accuracy.

AMS subject classifications: 76D07, 76M25, 65N12, 65N15, 35B45

Key words: Stokes' equations, deep neural network method, first-order velocity-pressure-vorticity formulation.

1 Stokes' equations

Recently, the deep neural network (DNN) methods have attracted remarkable attention in the field of computational fluid dynamics [9,33,57,61]. In contrast to classical methods

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such as finite element, finite difference, and finite volume, the DNN approach does not require a mesh topology and can achieve good accuracy even when the domain of interest is presented by scattered discrete points [50]. Moreover, it requires less number of parameters to achieve the same accuracy as with finite element method on the grid points [37]. And it can lessen or even overcome the curse of dimensionality for high-dimensional problems [20], it also has great potential in nonlinear approximations [51, 55]. However, the performance of existing DNN methods may degrade for the Navier-Stokes equations with low regularity, e.g. when sharp local gradients present in a broad computational domain [39, 48]. Noting that most DNN methods such as the recently developed physics informed neural network (PINN) method [33] employs the residual of equations as the loss function, which requires derivatives of variables that is only applicable to problems whose solutions are sufficiently smooth, for example $(\boldsymbol{u},p)$ at least in $[H^2(\Omega)]^2\times H^1(\Omega)$ for the Stokes equations, and is not able to cope properly with problems that have non-smooth solutions. As exactly solving the Stokes's equations with low regularity is the preliminary for many Navier-Stokes applications such as the lid driven cavity problems and other problems with singular sources and sharp interfaces [2, 15, 30], etc. In this paper, we propose a deep neural network method for the Stokes' equations with reduced regularity, i.e. $(\boldsymbol{u},p)\in[H^1(\Omega)]^2\times H^1(\Omega)$.

For simplicity, consider the following Stokes' problem which seeks a velocity field $\boldsymbol{u}$ and a pressure unknown p satisfying

$$\begin{cases} -\nu\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f}, & \text{in }\Omega,\\ \nabla\cdot\boldsymbol{u}=0, & \text{in }\Omega,\\ \boldsymbol{u}=\boldsymbol{g}, & \text{on }\Gamma, \end{cases} \tag{1.1}$$

where Ω is a bounded polygonal domain with boundary $\Gamma=\partial\Omega$ in d ($d=2$ or 3) dimension, $\nu>0$ is a constant viscosity parameter, $\boldsymbol{f}=(f^{(1)},\cdots,f^{(d)})$ represents external source, $\boldsymbol{g}$ is a given Dirichlet boundary condition, and the pressure p is assumed to have mean value zero; i.e., $\int_\Omega p d\Omega=0$. Noting that the classical solution of (1.1) satisfies $\boldsymbol{u}\in[C^2(\Omega)\cap C^0(\Omega)]^d$ and $p\in C^1(\Omega)$.

Various deep learning based methods have been proposed and explored recently for the simulation of partial differential equations, such as the least square methods [10, 17], the deep Ritz methods [19], the physics-informed neural network (PINN) methods [33, 51, 52] and the variational formulation based methods [34, 35] among many others [11, 28, 29, 42, 55]. Here we roughly classify these methods according to their formulation of the loss functions; the reader is referred to [4, 35] and the references therein for a more detailed review. The least square methods may trace back to the 1990s., Dissanayake and Phan-Thhien [17] have proposed training neural networks via a least square functional that based on the original formulation of the PDEs. Thereafter the method have been further developed and it has been shown that the sampling points can be obtained by a random sampling [26], which is beneficial for high dimensional problems. Despite these advantages, for a second-order PDE, the minimization of the loss