Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2021-0126

Vol. **31**, No. 3, pp. 771-815
March 2022

High-Order Positivity-Preserving Well-Balanced Discontinuous Galerkin Methods for Euler Equations with Gravitation on Unstructured Meshes

Weijie Zhang[1,2], Yulong Xing[2], Yinhua Xia[1] and Yan Xu[1,*]

[1] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.*
[2] *Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA.*

Received 10 June 2021; Accepted (in revised version) 19 October 2021

Abstract. In this paper, we propose a high-order accurate discontinuous Galerkin (DG) method for the compressible Euler equations under gravitational fields on unstructured meshes. The scheme preserves a general hydrostatic equilibrium state and provably guarantees the positivity of density and pressure at the same time. Comparing with the work on the well-balanced scheme for Euler equations with gravitation on rectangular meshes, the extension to triangular meshes is conceptually plausible but highly nontrivial. We first introduce a special way to recover the equilibrium state and then design a group of novel variables at the interface of two adjacent cells, which plays an important role in the well-balanced and positivity-preserving properties. One main challenge is that the well-balanced schemes may not have the weak positivity property. In order to achieve the well-balanced and positivity-preserving properties simultaneously while maintaining high-order accuracy, we carefully design DG spatial discretization with well-balanced numerical fluxes and suitable source term approximation. For the ideal gas, we prove that the resulting well-balanced scheme, coupled with strong stability preserving time discretizations, satisfies a weak positivity property. A simple existing limiter can be applied to enforce the positivity-preserving property, without losing high-order accuracy and conservation. Extensive one- and two-dimensional numerical examples demonstrate the desired properties of the proposed scheme, as well as its high resolution and robustness.

AMS subject classifications: 65M60, 35L60, 35L65, 65M12

Key words: Discontinuous Galerkin methods, Euler equations, gravitational field, positivity-preserving, well-balanced, triangular mesh.

*Corresponding author. *Email addresses:* `zhwj@mail.ustc.edu.cn` (W. Zhang), `xing.205@osu.edu` (Y. Xing), `yhxia@ustc.edu.cn` (Y. Xia), `yxu@ustc.edu.cn` (Y. Xu)

1 Introduction

In this paper, we develop high-order accurate positivity-preserving well-balanced discontinuous Galerkin (DG) methods for the compressible Euler equations with gravitation on unstructured meshes. This model has many interesting astrophysical and atmospheric applications, and takes the form

$$\boldsymbol{U}_t+\nabla\cdot\boldsymbol{F}(\boldsymbol{U})=\boldsymbol{S}(\boldsymbol{U},\nabla\phi), \tag{1.1}$$

in the d-dimensional case, with

$$\boldsymbol{U}=\begin{pmatrix}\rho\\ \boldsymbol{m}\\ E\end{pmatrix},\quad \boldsymbol{F}(\boldsymbol{U})=\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\boldsymbol{u}\otimes\boldsymbol{u}+p\boldsymbol{I}_d\\ (E+p)\boldsymbol{u}\end{pmatrix},\quad \boldsymbol{S}(\boldsymbol{U},\nabla\phi)=\begin{pmatrix}0\\ -\rho\nabla\phi\\ -\boldsymbol{m}\cdot\nabla\phi\end{pmatrix}, \tag{1.2}$$

where $\boldsymbol{m}=\rho\boldsymbol{u}$ denotes the momentum vector; $\rho,\boldsymbol{u},p$ denote the fluid density, velocity and pressure; $\boldsymbol{I}_d$ is the identity matrix of size d; $E=\frac{1}{2}\rho\|\boldsymbol{u}\|^2+\rho e$ (e is specific internal energy) is the non-gravitational energy; $\phi(\boldsymbol{x})$ is the time independent gravitational potential. The pressure p is linked to the density ρ and the internal energy e. In this paper, we consider the ideal gas with the following equation of state

$$p=(\gamma-1)\rho e=(\gamma-1)\left(E-\frac{\|\boldsymbol{m}\|^2}{2\rho}\right), \tag{1.3}$$

where γ is the ratio of specific heats.

The Euler equations under gravitation fields admit hydrostatic equilibrium solutions, in which the gravitational source term is exactly balanced by the flux gradient. Two well-known hydrostatic equilibrium states are the isothermal and the polytropic equilibria. For these hyperbolic balance laws, the well-balanced schemes are introduced to exactly preserve the equilibrium states at a discrete level. An important advantage of the well-balanced schemes is that they can effectively and accurately resolve the nearly equilibrium flows on relatively coarse meshes. These nearly equilibrium flows are small perturbation of the hydrostatic equilibrium states, and often appear in the astrophysical and atmospheric applications. Well-balanced methods for the shallow water equations with source term have been extensively studied in the past few decades, see [1,4,7,18,21, 29,42,44] and the review papers [28,46]. Recently, research on well-balanced schemes for the Euler equations with gravitation has attracted much attention. Such method was first studied by LeVeque and Bale [30] based on the quasi-steady wave propagation methods designed for the shallow water equations. After that, extensive well-balanced methods for the Euler equations with gravitation have been designed and studied within the framework of finite volume [2,3,5,8,9,11,17,22,25,27,31,38], finite difference [19,26,33,45], finite element discontinuous Galerkin (DG) [10, 32, 34, 39] and gas-kinetic schemes [47], and both first order and high order methods have been investigated. Recently, there are increasing interests in designing well-balanced methods for general equilibrium state.