Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2021-0201

Vol. **31**, No. 4, pp. 1162-1179
April 2022

Deep Unfitted Nitsche Method for Elliptic Interface Problems

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Received 15 October 2021; Accepted (in revised version) 9 January 2022

Abstract. This paper proposes a deep unfitted Nitsche method for solving elliptic interface problems with high contrasts in high dimensions. To capture discontinuities of the solution caused by interfaces, we reformulate the problem as an energy minimization problem involving two weakly coupled components. This enables us to train two deep neural networks to represent two components of the solution in high-dimensional space. The curse of dimensionality is alleviated by using the Monte-Carlo method to discretize the unfitted Nitsche energy functional. We present several numerical examples to show the performance of the proposed method.

AMS subject classifications: 78M10, 78A48, 47A70, 35P99
Key words: Deep learning, unfitted Nitsche method, interface problem, deep neural network.

1 Introduction

In this paper, we continue our previous studies on elliptic interface problems [11, 14, 15], arising in many applications such as fluid dynamics and materials science, where the background consists of rather different materials on the subdomains separated by smooth curves (or surfaces) called interfaces. We aim to address the high-dimensional challenge, which is well known as the curse of dimensionality leading to unaffordable computational time in traditional numerical methods (*e.g.*, finite difference and finite element methods).

Deep neural networks have been shown as a powerful tool to overcome the curse of dimensionality [4, 6, 9, 37], and have been applied to solve partial differential equations (PDEs), *e.g.*, the deep BSDE method [8, 16], the deep Galerkin method (DGM) [33],

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the physics-informed neural networks (PINNs) [31], the deep Ritz method (DRM) [7], and the weak adversarial networks (WAN) [38]. The deep BSDE reformulates the time-dependent equations into stochastic optimization problems. DGM and PINNs train neural networks by minimizing the mean squared error loss of the equation residual, while DRM trains networks by minimizing the energy functional of the variational problem equivalent to the PDEs. WAN uses the weak formulation and trains the primary and adversarial network alternatively using the min-max weak formulation. Moreover, the convergence of DRM was studied in [5, 27], and the deep Nitsche method was proposed in [26], which enhanced the deep Ritz method with natural treatment of essential boundary conditions. In a recent work [32], Sheng and Yang trained an additional neural network to impose the Dirichlet boundary conditions. However, in general, these neural network-based methods require the smoothness of the solutions to the PDEs. They thus can not be directly used to solve the elliptic interface problems, where the solutions are only piecewise smooth.

In literature, there are some recent works of solving elliptic interface problems using neural networks. For example, [36] proposed a network architecture similar to the deep Ritz method [7], and solved the equivalent variational problem with the boundary conditions approximated by a shallow neural network. [19] used different neural networks to approximate the solutions in disjoint subdomains. They reformulated the interface problem as a least-squares problem and solved it by stochastic gradient descent. [23] proposed the discontinuity capturing shallow neural network (DCSNN) to approximate piecewise continuous functions and solved elliptic interface problems by minimizing the mean squared error loss consisting of the residual of the equation, boundary and interface jump conditions.

In this paper, we propose a deep learning method for interface problems based on the minimization of the unfitted Nitsche energy functional, inspired by our previous studies [11–13] on the unfitted Nitsche method. One of the most significant differences between the unfitted Nitsche method [2, 12, 13, 17, 21] and other numerical methods for interface problems (*e.g.*, the immersed type numerical methods [25, 34, 35, 39]) is that the unfitted Nitsche finite element functions can be discontinuous inside elements. This is possible by adopting two different sets of basis functions on the interface elements (*i.e.*, the elements cut by the interface) which are weakly coupled together using Nitsche methods. Based on the unfitted Nitsche formulation, we can define the so-call unfitted Nitsche energy functional (see equation (2.11)). It turns out that the weak formulation of unfitted Nitsche method is just the Euler-Lagrange equation of unfitted Nitsche energy functional. To address the challenges of the curse of dimensionality, we naturally use deep neural networks to represent functions in high dimensions. Following the idea of classical unfitted Nitsche method [2, 12, 13, 17], we use two deep neural networks: one for the part inside the interface and the other one for the region outside the interface. These two parts are weakly connected using Nitsche method. The deep unfitted Nitsche method trains the two neural network functions independently using the same unfitted Nitsche energy functional.