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Numerical Integrators for Dispersion-Managed KdV Equation

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Abstract. In this paper, we consider the numerics of the dispersion-managed Korteweg-de Vries (DM-KdV) equation for describing wave propagations in inhomogeneous media. The DM-KdV equation contains a variable dispersion map with discontinuity, which makes the solution non-smooth in time. We formally analyze the convergence order reduction problems of some popular numerical methods including finite difference and time-splitting for solving the DM-KdV equation, where a necessary constraint on the time step has been identified. Then, two exponential-type dispersion-map integrators up to second order accuracy are derived, which are efficiently incorporated with the Fourier pseudospectral discretization in space, and they can converge regardless the discontinuity and the step size. Numerical comparisons show the advantage of the proposed methods with the application to solitary wave dynamics and extension to the fast & strong dispersion-management regime.

AMS subject classifications: 65M12, 65M15, 65M70, 65Z05, 35Q53

Key words: KdV equation, dispersion management, discontinuous coefficient, convergence order, finite difference, time-splitting, exponential integrator, pseudospectral method.

1 Introduction

The Korteweg-de Vries (KdV) equation which in classical form reads

$$\partial_t w - \partial_x^3 w + \partial_x w^2 = 0, \tag{1.1}$$

is of paramount importance in mathematical and physical studies, ever since the discovery of the existence of solitary wave (soliton) solutions. To enhance the dynamics of soliton streams, the technique of dispersion management has widely applied to many

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physical models [1,7,11,12,29,35,37,39,40,47]. In this work, we consider the following dispersion-managed KdV (DM-KdV) equation [7]:

$$\begin{cases} \partial_t u(x,t)+\gamma(t)\partial_x^3 u(x,t)+\partial_x(u(x,t))^2=0, & t>0, \quad x\in\mathbb{T}, \\ u(x,0)=u_0(x), & x\in\mathbb{T}, \end{cases} \tag{1.2}$$

where $\mathbb{T}$ is a torus (periodic boundary condition), $u=u(x,t):\mathbb{T}\times[0,\infty)\to\mathbb{R}$ is the unknown, $u_0(x):\mathbb{T}\to\mathbb{R}$ is the given initial data, and $\gamma(t):[0,\infty)\to\mathbb{R}$ is the dispersion map:

$$\gamma(t):=\begin{cases} -1, & (m-1)p_0\le t<mp_0-\frac{p_0}{2}, \\ 1, & mp_0-\frac{p_0}{2}\le t<mp_0, \end{cases} \qquad \text{for } m\in\mathbb{N}_+, \tag{1.3}$$

i.e., $\gamma(t)$ is a step function periodically extended with some period $p_0>0$. The DM-KdV (1.2) can be used to describe in general the gravity-capillary waves propagating in a periodically inhomogeneous waveguide [7,11]. The coefficient (1.3) is the so-called dispersion management/dispersion map that frequently occurred in fiber-optic communications, and its stabilization effect on the dispersion-managed nonlinear Schrödinger (DM-NLS) model has drawn wide attentions [2,12,29,35] due to the essential role of NLS in optics. Such stabilization of the dispersion management on the KdV equation which is a mathematical model as important as NLS [43], is also very attractive and [7] made the effort to study DM-KdV by asymptotics and numerics.

Theoretically and numerically, the constant-coefficient KdV-type equations (1.1) have been extensively studied in the literature, including the mathematical well-posedness theory [8,24,25] and different kinds of numerical approximations, such as finite difference [9,23], time-splitting [16,17,41], exponential integrator [20,32], spectral discretization [34, 46] and discontinuous Galerkin method [3,4,10,26–28,46]. The aforementioned classical numerical methods have been analyzed and applied in cases when the solution of (1.1) is smooth enough. For a comparison of them, we refer to the work [38]. Some recent efforts have been made to consider the numerical solution of (1.1) when some non-smoothness is introduced to the spatial direction of the solution through the initial data [20,44,45]. So far, the whole counterpart studies for the DM-KdV equation (1.2) are still missing to our best knowledge. However, the study of soliton dynamics in the DM-KdV equation rely on efficient and accurate numerical simulations [7].

Mathematically, the DM-KdV equation (1.2) with (1.3) can be viewed as consecutive compositions of two classical constant-coefficient KdV flows. Thus, the global well-posedness of the classical KdV equation [24] ensures the local well-posedness of (1.2). The rigorous analysis of such type of problems has been made for DM-NLS [2], but is yet to be done on the KdV model. In this paper, we begin with investigations of the exact solution of (1.2) in the physical space and Fourier space, where our numerical simulations in fact suggest that (1.2) shares the same well-posedness theory with the classical model (1.1) and the solution in space preserves the smoothness of the initial data.

However numerically, the discontinuities in the dispersion coefficient (1.3) induce non-smoothness to the time direction of the solution. In fact, as we shall discuss later