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Random Batch Particle Methods for the Homogeneous Landau Equation

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Abstract. We consider in this paper random batch particle methods for efficiently solving the homogeneous Landau equation in plasma physics. The methods are stochastic variations of the particle methods proposed by Carrillo et al. [J. Comput. Phys.: X 7: 100066, 2020] using the random batch strategy. The collisions only take place inside the small but randomly selected batches so that the computational cost is reduced to $\mathcal{O}(N)$ per time step. Meanwhile, our methods can preserve the conservation of mass, momentum, energy and the decay of entropy. Several numerical examples are performed to validate our methods.

AMS subject classifications: 65C35, 65Y20, 82C40, 82D10

Key words: Homogeneous Landau equation, random batch particle method, Coulomb collision.

1 Introduction

The Fokker-Planck-Landau equation, originally derived by Landau [28], is a fundamental integro-differential equation describing the evolution of the distribution for charged particles in plasma physics [37]. It models the binary collisions between charged particles with long-range Coulomb interaction, which is the grazing limit of the Boltzmann equation [12,14,41]. Denote by $f(t,x,v)$ the mass distribution of charged particles at time t, position x with velocity v, the Fokker-Planck-Landau equation is

$$\partial_t f + v\cdot\nabla_x f + F\cdot\nabla_v f = Q(f,f) \tag{1.1}$$

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with the Landau collision operator

$$Q(f,f)=\nabla_v\cdot\left(\int_{\mathbb{R}^d}A(v-v_*)\left(f(v_*)\nabla_v f(v)-f(v)\nabla_{v_*}f(v_*)\right)\mathrm{d}v_*\right). \tag{1.2}$$

Eq. (1.1) is a mean-field kinetic equation. The left-hand side is the Vlasov equation modeling the transport of charged particles, where F is the acceleration due to external or self-consistent forces under the effects of electrostatic and magnetic fields. The Landau collision operator $Q(f,f)$ describes the binary collisions between charged particles of single species with long-range Coulomb interactions. The collision kernel

$$A(z)=\Lambda|z|^\gamma(|z|^2I_d-z\otimes z),\quad -d-1\le\gamma\le 1,\quad \Lambda>0,\quad d\ge 2,$$

is symmetric positive semi-definite, $A(z)=A(-z)$, $\ker A(z)=\mathbb{R}z$. Similar to the Boltzmann equation, $\gamma>0$, $\gamma=0$, $\gamma<0$ represents the hard potential, Maxwell molecules and soft potential case respectively. The Coulomb potential where $d=3$, $\gamma=-3$ is of great significance since it is relevant in physical plasma applications [12].

In the numerical aspect of Eq. (1.1), the approximation of the nonlocal quadratic Landau collision operator $Q(f,f)$ is a major difficulty. Therefore, in this paper, we only focus on the spatially homogeneous Landau equation

$$\partial_t f=Q(f,f). \tag{1.3}$$

It is well-known that Eq. (1.3) has conservation of mass, momentum and energy since $\int_{\mathbb{R}^d}Q(f,f)(1,v,|v|^2)\mathrm{d}v=\mathbf{0}$. The Boltzmann entropy

$$E(f)=\int_{\mathbb{R}^d}f\log f\,\mathrm{d}v$$

is dissipated through

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-D=-\frac{1}{2}\iint_{\mathbb{R}^{2d}}B_{v,v_*}\cdot A(v-v_*)B_{v,v_*}ff_*\,\mathrm{d}v\mathrm{d}v_*\le 0.$$

Here,

$$B_{v,v_*}=\nabla_v\frac{\delta E}{\delta f}-\nabla_{v_*}\frac{\delta E_*}{\delta f_*},\quad \frac{\delta E}{\delta f}=\log f+1,$$

and $f_*=f(v_*)$, $E_*=E(f_*)$ for short. Moreover, f is the equilibrium of (1.3) if and only if f is given by the Maxwellian

$$\mathcal{M}_{\rho,u,T}=\frac{\rho}{(2\pi T)^{d/2}}\exp\left(-\frac{|v-u|^2}{2T}\right) \tag{1.4}$$

with ρ (density), u (velocity), T (temperature) determined from the conserved mass, momentum and total energy defined respectively by

$$\rho=\int_{\mathbb{R}^d}f\mathrm{d}v,\quad \rho u=\int_{\mathbb{R}^d}vf\mathrm{d}v,\quad \rho u^2+\rho dT=\int_{\mathbb{R}^d}|v|^2f\mathrm{d}v.$$