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An Efficient Iterative Method for the Formulation of Flow Networks

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Abstract. We propose an efficient iterative convolution thresholding method for the formulation of flow networks where the fluid is modeled by the Darcy–Stokes flow with the presence of volume sources. The method is based on the minimization of the dissipation power in the fluid region with a Darcy term. The flow network is represented by its characteristic function and the energy is approximated under this representation. The minimization problem can then be approximately solved by alternating: 1) solving a Brinkman equation to model the Darcy–Stokes flow and 2) updating the characteristic function by a simple convolution and thresholding step. The proposed method is simple and easy to implement. We prove mathematically that the iterative method has the total energy decaying property. Numerical experiments demonstrate the performance and robustness of the proposed method and interesting structures are observed.

AMS subject classifications: 35K08, 35Q35, 49Q10, 76S05

Key words: Topology optimization, convolution, thresholding, Darcy–Stokes flow.

1 Introduction

The flow networks are the cornerstone infrastructure of transporting substances and information. They are classified into two categories, social transport networks [1] and biological transport networks [2], based on whether they are created and designed by humans. Examples of social transport networks include but not limited to electrical powergrids, railway and highway networks, information networks such as the internet, and supply and demand networks. Biological transport networks are widely present in the

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organism, *e.g.*leaf venation, blood vessel systems, neural networks *etc*. The primary design purpose of a flow network is to balance the cost of constructing and maintaining the structure, the efficiency of transportation, and resilience to local failure and disturbance.

Explosive growing demand for transferring matter and messages put urgent needs in studies on transport networks. Material properties are closely related to their structure. Studies on networks' structures not only help us build efficient and robust networks but also shed insights into the understanding of biological transport networks where natural selection plays the role of optimization processes. For example, the cubic laws for blood vessel which was discovered in experiments can be explained by balancing the delivering energy cost of blood flow and the metabolic cost of blood cells [3,4].

The difficulties and challenges of designing flow networks are the non-convex optimization property and ill-posedness (no solution, *e.g.*Steiner tree model [5] or multiple solutions [6–8]) of the formulated mathematical models. It is necessary to adopt filtering techniques and/or design suitable objective functionals to resolve the well-posedness. To tackle the non-convexity, stability, convergence and ability to avoid local optima are critical for a practical numerical algorithm.

There are many algorithms developed for constructing transport networks. In [2], discrete adaption dynamics, which reacts to local stimulus for network formation, is proposed. The continuous analogy has also been studied and analyzed [9–12]. Tree-like structures have been observed using their models. They also discover that loops emerge when strong fluctuations of flow sources present. In [13], a flow network system consists of Darcy and Stokes region is analyzed, and optimal layout is obtained through sensitivity analysis and the first-order method MMA [14]. Many other methods have also been developed in the context of shape/topology/structure optimization: density interpolation [15]; level-set method approach [16,17]; topological derivatives [18]; phase field method [19]; evolutionary structural optimization (ESO) approaches [20], and several others. To ensure well-posedness and mesh-independent solutions, regularization is usually needed in topology optimization. Many robust regularization approaches were introduced. For example, regularizing the optimization problem by introducing fictitious interface energy [21] or penalizing the original objective function by perimeter control functions based on phase field methods [22].

In this paper, we formulate the formation of flow networks as a constrained optimization problem. To be specific, we assume the fluid in the network is either a Stokes flow or governed by Darcy's law. The interface is implicitly represented by characteristic functions of corresponding domains and the network is formed by minimizing the total potential energy of the system or the dissipation of the fluids under volume constraints. It is motivated by the MBO method for approximating mean curvature flow using characteristic functions [23] and Esedoglu and Otto's [24] novel interpretation using minimizing movement for problems of multiphase flow with arbitrary surface tensions. The MBO method has attracted much attention due to its simplicity and unconditional stability. It has subsequently been extended to deal with many other applications, including