

A Continuous Finite Element Method with Homotopy Vanishing Viscosity for Solving the Static Eikonal Equation

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Abstract. We develop a second-order continuous finite element method for solving the static Eikonal equation. It is based on the vanishing viscosity approach with a homotopy method for solving the discretized nonlinear system. More specifically, the homotopy method is utilized to decrease the viscosity coefficient gradually, while Newton's method is applied to compute the solution for each viscosity coefficient. Newton's method alone converges for just big enough viscosity coefficients on very coarse grids and for simple 1D examples, but the proposed method is much more robust and guarantees the convergence of the nonlinear solver for all viscosity coefficients and for all examples over all grids. Numerical experiments from 1D to 3D are presented to confirm the second-order convergence and the effectiveness of the proposed method on both structured or unstructured meshes.

AMS subject classifications: 65N06, 65N12, 65N15

Key words: Eikonal equation, finite element method, homotopy method.

1 Introduction

We consider the static Eikonal equation

$$\begin{cases} |\nabla u(x)| = f(x), & x \in \Omega \setminus \Gamma, \\ u(x) = g(x), & x \in \Gamma, \end{cases} \quad (1.1)$$

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over the domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, where $f \geq 0$ and g are the given functions, Γ is the “boundary” which is a subset of $\bar{\Omega}$. Moreover, g is assumed to satisfy the compatible condition to guarantee the existence of the physical or viscosity solution of (1.1), see e.g. [29, 31].

Eikonal equation (1.1) has many applications such as computational fluid dynamics, optics, wave propagation, material science, differential geometry (geodesics), image processing and computer graphics [1, 5, 9, 11, 40, 41, 46]. When solving two-phases flow problems [12], the solution u of Eq. (1.1) with $f \equiv 1$ and $g \equiv 0$ is the distance function with Γ as its 0-level set, which makes u be able to track the interface between two phases. In the so-called Shape-from-Shading problem [34], the solution of (1.1) in 2D reconstructs the surface $z = u(x, y)$ based on $f(x, y)$ which is related to the brightness $I(x, y)$ of the surface under a remote vertical light source as $f = \sqrt{1 - I(x, y)^2} / I(x, y)$. In the seismic ray method where the separation of variable is utilized to locate the high-frequency seismic body wave in the media [37], the solution u of (1.1) describes the travel time with $f(x) = 1/v(x)$ as the slowness associated to the velocity $v(x)$ in the media [6].

Mathematically speaking, Eikonal equation (1.1) is a typical example of Hamilton-Jacobi equation $H(x, u, \nabla u) = f$ by taking $H(x, u, \nabla u) = |\nabla u|$. Thus the difficulties of solving Hamilton-Jacobi equation such as the nonlinearity and non-uniqueness apply to the Eikonal equation (1.1). Therefore, the generalized solution or *viscosity solution* has to be sought for solving the Eikonal equation (1.1) [8–11, 29, 31]. The concept of viscosity solution is reasonable and satisfactory since: 1) if u is a smooth solution of (1.1), then it is a viscosity solution; 2) if the viscosity solution u is differentiable at some point, then it satisfies the equality (1.1); 3) the viscosity solution is unique given appropriate boundary condition; 4) the solution obtained by the vanishing viscosity method is the viscosity solution.

There are many numerical methods to solve the Eikonal equation and to compute the viscosity solution. The characteristic method has been developed to solve (1.1) by solving a first-order ODE [25]. However, the method is hard to find the global solution and has to deal with the coupling of the spatial variables and the phase space variables. The level set formulation is utilized to introduce a time variable to solve the static Eikonal equation [28, 32, 33]. A monotone finite difference method and a vanishing viscosity method are used to solve the time-dependent Cauchy problem of the Hamilton-Jacobi equation with the form of $H(\nabla u)$ in [11], where the convergence rate is obtained explicitly. The fast marching method and fast sweeping method coupled with finite difference discretization are developed to solve the Eikonal equation. The fast marching method is based on entropy-satisfying upwind schemes and fast sorting techniques where the solution is updated by sequentially following the causality [41, 44]. The fast sweeping method does not need heap-sort and the updating follows the causality along with the characteristics in a parallel way [26, 27, 46–48]. There are several approaches based on the finite element method for solving Eikonal equation. For example, a continuous finite element method based on minimization of the residual in L^1 norm is proposed to solve stationary Hamilton-Jacobi equations [18–20]; a discontinuous Galerkin method based