Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2021-0255

Vol. **31**, No. 5, pp. 1434-1466
May 2022

A Mixed Finite Element Scheme for Biharmonic Equation with Variable Coefficient and von Kármán Equations

Huangxin Chen[1], Amiya K. Pani[2] and Weifeng Qiu[3,*]

[1] *School of Mathematical Sciences and Fujian Provincial Key Laboratory on Mathematical Modeling and High Performance Scientific Computing, Xiamen University, Fujian, 361005, China.*
[2] *Department of Mathematics, Indian Institute of Technology, Bombay, Powai, Mumbai-400076, India.*
[3] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong, China.*

Received 28 December 2021; Accepted (in revised version) 10 April 2022

Abstract. In this paper, a new mixed finite element scheme using element-wise stabilization is introduced for the biharmonic equation with variable coefficient on Lipschitz polyhedral domains. The proposed scheme doesn't involve any integration along mesh interfaces. The gradient of the solution is approximated by $H(\mathrm{div})$-conforming BDM_{k+1} element or vector valued Lagrange element with order $k+1$, while the solution is approximated by Lagrange element with order $k+2$ for any $k\geq 0$. This scheme can be easily implemented and produces symmetric and positive definite linear system. We provide a new discrete H^2-norm stability, which is useful not only in analysis of this scheme but also in C^0 interior penalty methods and DG methods. Optimal convergences in both discrete H^2-norm and L^2-norm are derived. This scheme with its analysis is further generalized to the von Kármán equations. Finally, numerical results verifying the theoretical estimates of the proposed algorithms are also presented.

AMS subject classifications: 65N12, 65N30

Key words: Biharmonic equation, von Kármán equations, mixed finite element methods, element-wise stabilization, discrete H2-stability, positive definite.

1 Introduction

In the first part of this paper, a new mixed finite element scheme is proposed and analyzed for the following biharmonic equation with variable coefficient:

*Corresponding author. *Email addresses:* `chx@xmu.edu.cn` (H. Chen), `akp@math.iitb.ac.in` (A. K. Pani), `weifeqiu@cityu.edu.hk` (W. Qiu)

$$
\begin{aligned}
\Delta(\kappa\Delta u) &= f, \quad \text{in } \Omega, && (1.1a)\\
u &= 0, \quad \text{on } \partial\Omega, && (1.1b)\\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial\Omega, && (1.1c)
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^d (d \in \mathbb{N})$ is a Lipschitz polygonal or polyhedral domain, the coefficient $\kappa \in W^{1,\infty}(\Omega)$ such that $0 < \kappa_0 \le \kappa(\boldsymbol{x}) \le \kappa_1$, and $f \in H^{-1}(\Omega)$. By using element-wise stabilization, our scheme doesn't involve any integration along mesh interfaces. Our scheme uses $H(\text{div})$-conforming BDM_{k+1} or vector valued Lagrange element with order $k+1$ to approximate $\boldsymbol{w} = \nabla u$, and approximates u with Lagrange element with order $k+2$ for any $k \ge 0$. The biharmonic equation with variable coefficient (1.1) has some applications in practical problems such as bending problems of elastic plates with variable thickness [5], the fourth-order Cahn-Hilliard equation in two or three dimension with κ as the phenomenological mobility coefficient [25] in material science, etc.

The second part of this paper is related to an application of our scheme to the von Kármán model, which can be stated as follows:

$$
\begin{aligned}
\Delta^2\xi - [\xi,\psi] &= f, \quad \text{in } \Omega, && (1.2a)\\
\Delta^2\psi + [\xi,\xi] &= 0, \quad \text{in } \Omega, && (1.2b)\\
\xi = \frac{\partial \xi}{\partial n} &= 0, \quad \text{on } \partial\Omega, && (1.2c)\\
\psi = \frac{\partial \psi}{\partial n} &= 0, \quad \text{on } \partial\Omega, && (1.2d)
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^2$ is a Lipschitz polygonal domain, $f \in H^{-1}(\Omega)$, and the von Kármán bracket $[\cdot,\cdot]$ appearing in (1.2a) and (1.2b) is defined by

$$
[\eta,\phi] = \frac{\partial^2\eta}{\partial x_1^2}\frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\eta}{\partial x_2^2}\frac{\partial^2\phi}{\partial x_1^2} - 2\frac{\partial^2\eta}{\partial x_1\partial x_2}\frac{\partial^2\phi}{\partial x_1\partial x_2} = \text{cof}(D^2\eta) : D^2\phi.
$$

Here, $\text{cof}(D^2\eta)$ denotes the cofactor matrix of the Hessian of η and $A:B$ denotes the Frobenius inner product of the matrices A and B.

In literature, there are many numerical methods available for the biharmonic equation, that is, the problem (1.1) with $\kappa = 1$. Some of them can be easily generalized to include biharmonic problem with variable coefficients. We provide below a brief summary of results which are relevant to our present investigation.

- *Numerical methods approximating both u and Δu.* The Ciarlet and Raviart (C-R) method [16] uses u and Δu as unknowns and thereby, gives rise to a system of Poisson problems. Then, H^1-conforming finite element spaces are used to approximate both u and Δu, and it has no stabilization along mesh interfaces. Thus, the C-R method