Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2021-0193

Vol. **31**, No. 5, pp. 1467-1488
May 2022

Eulerian Algorithms for Computing the Forward Finite Time Lyapunov Exponent Without Finite Difference Upon the Flow Map

Guoqiao You[1], Changfeng Xue[2,*] and Shaozhong Deng[3]

[1] *School of Statistics and Data Science, Nanjing Audit University, Nanjing, Jiangsu 211815, P.R. China.*
[2] *School of Mathematics and Physics, Yancheng Institute of Technology, Yancheng, Jiangsu 224051, P.R. China.*
[3] *Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223-0001, USA.*

Received 28 September 2021; Accepted (in revised version) 12 April 2022

Abstract. In this paper, effective Eulerian algorithms are introduced for the computation of the *forward* finite time Lyapunov exponent (FTLE) of smooth flow fields. The advantages of the proposed algorithms mainly manifest in two aspects. First, previous Eulerian approaches for computing the FTLE field are improved so that the Jacobian of the flow map can be obtained by directly solving a corresponding system of partial differential equations, rather than by implementing certain finite difference upon the flow map, which can significantly improve the accuracy of the numerical solution especially near the FTLE ridges. Second, in the proposed algorithms, all computations are done *on the fly*, that is, all required partial differential equations are solved *forward* in time, which is practically more natural. The new algorithms still maintain the optimal computational complexity as well as the second order accuracy. Numerical examples demonstrate the effectiveness of the proposed algorithms.

AMS subject classifications: 37A25, 37M25, 76M27

Key words: Finite time Lyapunov exponent, flow map, flow visualization, dynamical system.

1 Introduction

Eulerian algorithms have been proposed [1–3] as alternatives to Lagrangian approaches for computing the finite time Lyapunov exponent (FTLE) [4–8] in the numerical study

*Corresponding author. *Email addresses:* `magqyou@nau.edu.cn` (G. You), `cfxue@ycit.edu.cn` (C. Xue), `shaodeng@uncc.edu` (S. Deng)

 ©2022 Global-Science Press

of dynamical systems, where the evolution of particle trajectories is determined based on the level set method so that the flow map satisfies the Liouville equations. Two important issues, however, require further investigations. First, all these Eulerian methods have suggested to solve the underlying system of partial differential equations (PDEs) *backward* in time to obtain the *forward* flow map. That is, in order to obtain the *forward* flow map from the initial time $t=0$ to the final time $t=T$, one needs to solve the PDE system *backward* in time from $t=T$ to $t=0$. It is practically inconvenient, especially when the velocity data is obtained from some computational fluid dynamics solvers, because in this case we have to first store the whole velocity data at all time steps in the disk and then load it from the current time $t=T$ backward to the initial time. Second, in almost all existing algorithms, the flow map is first computed and a certain finite difference scheme is next used upon the computed flow map so as to obtain the Jacobian of the flow map and consequently the required FTLE field. Then the issue is that the numerical dissipation in typical finite difference schemes might cause non-negligible errors to the solution of the FTLE, especially near the FTLE ridges.

To address the first issue, Eulerian interpolation methods have been proposed to compute the forward flow map *on the fly* [9,10], namely, to compute the forward flow map, all PDEs are also solved in the forward direction as the flow map, but in these methods, the Jacobian of the flow map is still obtained by a finite difference scheme. On the other hand, to overcome the second issue, an Eulerian-type algorithm has been developed in [11] to more accurately compute the FTLE. In particular, a PDE system is first derived to govern the evolution of the Jacobian (rather than the flow map), and then the PDE system is solved to directly obtain the Jacobian and thus the FTLE field. In this approach, no finite difference is implemented upon the flow map and the resulting solution to the FTLE field is improved, especially near the FTLE ridges. Having said that, this approach has only addressed how to directly obtain the Jacobian of the backward flow map (referred to as the backward Jacobian hereafter) and thus the backward FTLE.

In the present paper, we propose Eulerian interpolation algorithms to compute the Jacobian of the forward flow map (referred to as the *forward* Jacobian hereafter) and the corresponding *forward* FTLE. We first derive a PDE system governing the evolution of the *forward* Jacobian and then solve this PDE system *forward* in time to directly obtain the forward Jacobian. At each intermediate time, the overall forward flow map and the forward Jacobian are constructed by using a standard interpolation routine such as the function `pchip/interp1/interp2/interp3/interpn` in MATLAB. The rest of the paper is organized as follows. First, some basic concepts and previous Eulerian approaches for computing the flow maps and the corresponding FTLE are briefly reviewed in Section 2. Then new Eulerian interpolation algorithms are proposed in Section 3. Next, complexity and accuracy analysis of the proposed algorithms is given in Section 4. Finally, numerical examples are presented in Section 5 to show the accuracy and effectiveness of the proposed algorithms.