

# Monotonicity Correction for the Finite Element Method of Anisotropic Diffusion Problems

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**Abstract.** We apply the monotonicity correction to the finite element method for the anisotropic diffusion problems, including linear and quadratic finite elements on triangular meshes. When formulating the finite element schemes, we need to calculate the integrals on every triangular element, whose results are the linear combination of the two-point pairs. Then we decompose the integral results into the main and remaining parts according to coefficient signs of two-point pairs. We apply the nonlinear correction to the positive remaining parts and move the negative remaining parts to the right side of the finite element equations. Finally, the original stiffness matrix can be transformed into a nonlinear M-matrix, and the corrected schemes have the positivity-preserving property. We also give the monotonicity correction to the time derivative term for the time-dependent problems. Numerical experiments show that the corrected finite element method has monotonicity and maintains the convergence order of the original schemes in  $H^1$ -norm and  $L^2$ -norm, respectively.

**AMS subject classifications:** 65N30

**Key words:** The finite element method, nonlinear M-matrix, monotonicity correction, positivity-preserving property, two-point pair.

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## 1 Introduction

In this paper, we consider the following diffusion problem

$$\begin{cases} -\nabla \cdot (A(x,y)\nabla u) = f, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma, \end{cases} \quad (1.1)$$

where

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- a).  $\Omega$  is an open-bounded, connected polygonal domain in  $\mathbf{R}^2$ , with boundary  $\Gamma = \partial\Omega$ .
- b).  $A(x,y) = (a_{ij}(x,y))_{i,j=1,2}$  is the diffusion tensor, whose entries are all Lipschitz continuous, and it is assumed to be symmetric, bounded, and uniformly positive definite on  $\Omega$ , i.e., there exist some  $0 < \gamma_1 < \gamma_2$ , such that

$$\gamma_1(v,v) \leq (A(x,y)v,v) \leq \gamma_2(v,v), \quad \forall v \in \mathbf{R}^2, \quad \forall (x,y) \in \overline{\Omega}. \quad (1.2)$$

- c). The source term  $f \in L^2(\Omega)$  is piecewise smooth and satisfies  $f \geq 0$ , boundary condition  $g \geq 0$ .

In practical applications, Eq. (1.1) is the prototype structure of complex problems, such as heat conduction, biological systems, plasma physics, and image processing. It could describe the variation of quality, density, or intensity of pressure. From the perspective of physics and the strong extremal principles in [1], these variables are nonnegative for Eq. (1.1) with boundary condition  $g \geq 0$ . The discrete maximum principle (DMP) forms an important qualitative property of second-order elliptic equations in scientific computing: if the source term is nonnegative, then the solution attains its minimum on the boundary, and the solution is everywhere nonnegative for the problem with nonnegative boundary data  $g$ . In general, since large source term variations, anisotropic diffusion and extremely distorted meshes, numerical solutions of discrete schemes without monotonicity usually carry non-physical information, which may result in the numerical solutions with negative values, and it is thus not preferred in the application. Therefore, the research of numerical schemes with these properties is significant.

It is well-known that the linear conforming finite element method (FEM) preserves DMP naturally [2]. However, the finite element method may be without the DMP-preserving property on some general meshes [3]. So scholars have studied many linear schemes with the above properties. A class of finite volume numerical schemes sharing both discrete conservation and discrete strong maximum principle have been proposed in [4]. Xu and Zikatanov [5] employ a special number treatment for convection terms so that they obtain a monotone scheme under some mild assumptions for finite element grids. In [6], Lu et al. apply a cutoff method in the computation of nonnegative solutions for anisotropic diffusion equations. In paper [7–11], authors present the sufficient conditions for DMP-preserving numerical schemes. Mudunuru and Nakshatrala [12] propose a robust computational framework that satisfies maximum principles. There are some results about DMP-preserving weak Galerkin methods under some grid constraints in [13, 14]. Richard Liska and Mikhail Shashkov [15] propose two approaches to enforce DMP: a posteriori correction of the discrete solution and the constrained optimization. Wang et al. [16] present two repair techniques for diamond schemes with DMP-preserving property of anisotropic diffusion problems. In regions of large source term variations, numerical schemes sometimes can cause the temperature to become negative, then algorithms based on slope limiters are proposed to fix this problem in [17].