

Extrapolation Cascadic Multigrid Method for Cell-Centered FV Discretization of Diffusion Equations with Strongly Discontinuous and Anisotropic Coefficients

Kejia Pan¹, Xiaoxin Wu¹, Yunlong Yu², Zhiqiang Sheng² and Guangwei Yuan^{2,*}

¹ School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, P.R. China.

² Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100088, P.R. China.

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Abstract. Extrapolation cascadic multigrid (EXCMG) method with conjugate gradient smoother is very efficient for solving the elliptic boundary value problems with linear finite element discretization. However, it is not trivial to generalize the vertex-centred EXCMG method to cell-centered finite volume (FV) methods for diffusion equations with strongly discontinuous and anisotropic coefficients, since a non-nested hierarchy of grid nodes are used in the cell-centered discretization. For cell-centered FV schemes, the vertex values (auxiliary unknowns) need to be approximated by cell-centered ones (primary unknowns). One of the novelties is to propose a new gradient transfer (GT) method of interpolating vertex unknowns with cell-centered ones, which is easy to implement and applicable to general diffusion tensors. The main novelty of this paper is to design a multigrid prolongation operator based on the GT method and splitting extrapolation method, and then propose a cell-centered EXCMG method with BiCGStab smoother for solving the large linear system resulting from linear FV discretization of diffusion equations with strongly discontinuous and anisotropic coefficients. Numerical experiments are presented to demonstrate the high efficiency of the proposed method.

AMS subject classifications: 65N55, 65N08

Key words: Diffusion equation, discontinuous coefficients, anisotropic coefficients, Richardson extrapolation, finite volume scheme, cell-centered multigrid method.

*Corresponding author. *Email addresses:* kejiapan@csu.edu.cn (K. Pan), 202101021@csu.edu.cn (X. Wu), yu_yunlong@iapcm.ac.cn (Y. Yu), sheng_zhiqiang@iapcm.ac.cn (S. Sheng), yuan_guangwei@iapcm.ac.cn (G. Yuan)

1 Introduction

Diffusion equations with discontinuous and anisotropic coefficients arise from many areas of applications. Examples of these problems include the flows through porous media with different porosity [1], the electric currents through material of different conductivities [2] and the solidification processes of materials [3].

Efficient and accurate numerical approaches for such problems are important areas of research. The emphasis in this paper is on multigrid (MG) method for solving the large scale diffusion problem especially when strongly discontinuous and anisotropic diffusion coefficients occur. Indeed, the discontinuity precludes standard MG methods, and some specific MG methods are proposed for such problems. In 1981, Alcouffe *et al.* [4] proposed the first MG method for two-dimensional diffusion equation with strongly discontinuous coefficients, in which diffusion coefficients are used in the design of interpolation operators. Then Behie *et al.* [5] extended this method to three-dimensional problems. Noted that they were all based on vertex-centered discretization and grid coarsening is done point-wise. Wesseling *et al.* [6] developed the first cell-centered MG method for interface problems and proposed constant prolongation and restriction operators. Then Khalil *et al.* [7] gave a comparison between vertex-centered and cell-centered MG methods, which showed that the former were more robust and needed less iterations to assure convergence, but the latter used less storage because of the transfer operators being constant.

In 1992, a cell-centered full approximation MG scheme for two-dimensional elliptic equations with discontinuous coefficients was proposed by Liu *et al.* in [8], in which transfer interpolation operator based on effective area was designed and MG efficiency was achieved. In 1999, V-cycle and W-cycle MG methods for two-dimensional elliptic equations with piece-wise constant coefficients and continuous coefficients on triangular meshes were proposed by Kwak in [9] and [10], in which new prolongation operators with energy bound were presented. In 2000, interface preserving coarsening algorithms were presented for complex interface problems by Wan *et al.* in [11], thus only easy linear interpolation operator was needed. In 2004, a W-cycle MG method for two-dimensional elliptic equations with coefficients-dependent prolongation operator was proposed by Kwak in [12], in which energy bound of the prolongation operator was also discussed. In 2006, a V-cycle MG method for two-dimensional elliptic equations with highly discontinuous coefficients was proposed by Kwak in [13], in which different prolongation operators were also discussed. In 2008, uniform convergent MG methods for linear finite element approximation of second-order elliptic boundary value problems with strongly discontinuous coefficients were proposed by Xu in [14]. Cell-centered MG methods for elliptic problems on semi-structured triangular grids with constant coefficients and discontinuous coefficients were proposed by Salinas in [15] and [16], and then further investigated by local Fourier analysis in [17]. In 2015, a MG method for three-dimensional elliptic equations with anisotropic discontinuous coefficients is proposed by Zhukov in [18]. In 2017, some MG methods for smooth diffusion problems were discussed in [19–23].