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# Continuous Finite Element Subgrid Basis Functions for Discontinuous Galerkin Schemes on Unstructured Polygonal Voronoi Meshes

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**Abstract.** We propose a new high order accurate *nodal* discontinuous Galerkin (DG) method for the solution of nonlinear hyperbolic systems of partial differential equations (PDE) on unstructured polygonal Voronoi meshes. Rather than using classical *polynomials* of degree $N$ inside each element, in our new approach the discrete solution is represented by *piecewise continuous polynomials* of degree $N$ *within* each Voronoi element, using a *continuous finite element* basis defined on a subgrid inside each polygon. We call the resulting subgrid basis an *agglomerated finite element* (AFE) basis for the DG method on general polygons, since it is obtained by the agglomeration of the finite element basis functions associated with the subgrid triangles. The basis functions on each sub-triangle are defined, as usual, on a universal reference element, hence allowing to compute *universal* mass, flux and stiffness matrices for the subgrid triangles once and for all in a pre-processing stage for the reference element only. Consequently, the construction of an efficient *quadrature-free* algorithm is possible, despite the unstructured nature of the computational grid. High order of accuracy in time is achieved thanks to the ADER approach, making use of an element-local space-time Galerkin finite element predictor.

The novel schemes are carefully validated against a set of typical benchmark problems for the compressible Euler and Navier-Stokes equations. The numerical results have been checked with reference solutions available in literature and also systematically compared, in terms of computational efficiency and accuracy, with those obtained by the corresponding modal DG version of the scheme.

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# 1 Introduction

Nonlinear systems of hyperbolic conservation laws are used to model a wide range of phenomena in nature, covering different fields of applications in science and engineering. Due to the complex structure of the governing partial differential equations (PDE) and the non-linearity of the problem, analytical solutions are extremely rare and very difficult to be found. Therefore, a lot of research has been devoted to the design of numerical schemes that aim at solving nonlinear systems of evolutionary PDE. Suitable discretizations in both space and time have been investigated in the past decades, starting from the pioneering work on Godunov-type finite volume schemes [58, 87, 93]. In this approach the numerical solution is stored under the form of piecewise *constant* cell averages within each control volume of the computational mesh, thus requiring a spatial reconstruction procedure in order to obtain higher order schemes, and the time evolution is obtained either by using Runge-Kutta timestepping, or directly discretizing the integral form of the conservation law at the aid of a fully-discrete one-step method. Alternatively, discontinuous Galerkin (DG) finite element methods can be used for the spatial approximation of the numerical solution, that in this case is directly expressed through high order *polynomials* within each control volume, allowing jumps of the discrete solution across element boundaries, leading thus to a natural high order piecewise polynomial data representation. Thus, DG schemes do not need any reconstruction procedure, unlike high order finite volume solvers. These methods were first applied to neutron transport equations [75] and later extended to general nonlinear systems of hyperbolic conservation laws in one and multiple space dimensions [25–29].

In the DG framework, the numerical solution is represented globally by piecewise polynomials of degree up to $N$ using a polynomial expansion in terms of a set of suitable basis functions inside each element, that can be either of nodal or modal type. The *nodal approach* allows very efficient schemes to be formulated in terms of a piecewise polynomial approximation with a judicious choice of the nodes, like the Gauss-Lobatto nodes [46] or the Gauss-Legendre nodes [76]. Nodal basis functions are typically used either on Cartesian meshes or on unstructured grids composed of simplex control volumes, namely triangles in 2D and tetrahedra in 3D, but there exist also nodal basis functions for more general unstructured meshes, see, e.g. [54]. This restriction is mainly due to the preferable requirement of a reference element where the basis functions are uniquely defined and the physical control volume is mapped to. Hence, the nodal basis is universal and only the mapping between the physical and the reference coordinate system contains