Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2022-0030

Vol. **32**, No. 2, pp. 364-400
August 2022

Moment-Based Multi-Resolution HWENO Scheme for Hyperbolic Conservation Laws

Jiayin Li[1], Chi-Wang Shu[2] and Jianxian Qiu[3,*]

[1] *School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, P.R. China.*
[2] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*
[3] *School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling and High-Performance Scientific Computing, Xiamen University, Xiamen, Fujian 361005, P.R. China.*

Received 28 January 2022; Accepted (in revised version) 23 May 2022

Abstract. In this paper, a high-order moment-based multi-resolution Hermite weighted essentially non-oscillatory (HWENO) scheme is designed for hyperbolic conservation laws. The main idea of this scheme is derived from our previous work [J. Comput. Phys., 446 (2021) 110653], in which the integral averages of the function and its first order derivative are used to reconstruct both the function and its first order derivative values at the boundaries. However, in this paper, only the function values at the Gauss-Lobatto points in the one or two dimensional case need to be reconstructed by using the information of the zeroth and first order moments. In addition, an extra modification procedure is used to modify those first order moments in the troubled-cells, which leads to an improvement of stability and an enhancement of resolution near discontinuities. To obtain the same order of accuracy, the size of the stencil required by this moment-based multi-resolution HWENO scheme is still the same as the general HWENO scheme and is more compact than the general WENO scheme. Moreover, the linear weights are not unique and are independent of the node position, and the CFL number can still be 0.6 whether for the one or two dimensional case, which has to be 0.2 in the two dimensional case for other HWENO schemes. Extensive numerical examples are given to demonstrate the stability and resolution of such moment-based multi-resolution HWENO scheme.

AMS subject classifications: 65M60, 65M99, 35L65

Key words: Moment-based scheme, multi-resolution scheme, HWENO scheme, hyperbolic conservation laws, KXRCF troubled-cell indicator, HLLC-flux.

---

*Corresponding author. *Email addresses:* jiayin@stu.xmu.edu.cn (J. Li), chi-wang_shu@brown.edu (C.-W. Shu), jxqiu@xmu.edu.cn (J. Qiu)

1 Introduction

In this paper, a high-order moment-based multi-resolution Hermite weighted essentially non-oscillatory (HWENO) scheme is designed for hyperbolic conservation laws

$$\begin{cases} u_t+\nabla\cdot f(u)=0, \\ u(x_1,\cdots,x_d,0)=u_0(x_1,\cdots,x_d). \end{cases} \tag{1.1}$$

We concentrate our attention on the one and two dimensional cases ($d=1$ or 2), and in these cases we denote x_1 as x and x_2 as y.

Conservation laws arise from the physical law that the conservative quantity in any control body can change only due to the flux passing through its boundaries, which naturally hold for many fundamental physical quantities, such as the mass, momentum, energy and so on. Such conservation laws are widely used in a broad spectrum of disciplines where wave motion or advective transport is important: gas dynamics, acoustics, elastodynamics, optics, geophysics, and biomechanics, to name but a few.

The differential equation (1.1) can be derived from the integral equation by simple manipulations provided that the conservative quantity and its corresponding flux are sufficiently smooth. This proviso is important because in practice many interesting solutions are not smooth, but contain discontinuities such as shock waves. A fundamental feature of nonlinear conservation laws is that discontinuities can easily develop spontaneously even from smooth initial data, and must be dealt with carefully both mathematically and computationally. At a discontinuity in the conservative quantity, the differential equation does not hold in the classical sense and it is important to remember that the integral form of the conservation laws does continue to hold which is more fundamental. This is also why we choose conservative schemes, such as the finite volume method considered in this paper, which is based on the integral form of the conservation laws.

Since conservation laws have a very wide range of applications and it is almost impossible in general to get their exact solutions, many scholars have explored and proposed a series of numerical methods and are still trying to improve the performance of these algorithms. In 1994, Liu et al. proposed the first finite volume WENO scheme in [17], and then, in 1996, Jiang and Shu improved this WENO scheme to fifth order and to conservative finite difference formulation (which is more efficient in multi-dimensions), and gave a general definition of the smoothness indicators and nonlinear weights in [12]. The methodology of such WENO schemes is to use a nonlinear convex combination of all the candidate stencils to improve the order of accuracy in smooth regions without destroying the non-oscillatory behavior near discontinuities. This is also the difference of such WENO schemes from the ENO schemes in [10,22,23], which only choose the locally smoothest stencil automatically among all the central and biased spatial stencils. Thereafter, different kinds of WENO schemes have been developed in, e.g. [1–3, 6, 18, 28, 29]. Although these WENO schemes work well for most of the problems we encountered, there is still room for improvement. For example, if we want to obtain a higher order