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A High-Order Direct Discontinuous Galerkin Method for Variable Density Incompressible Flows

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Abstract. In this work, we develop a novel high-order discontinuous Galerkin (DG) method for solving the incompressible Navier-Stokes equations with variable density. The incompressibility constraint at cell interfaces is relaxed by an artificial compressibility term. Then, since the hyperbolic nature of the governing equations is recovered, the simple and robust Harten-Lax-van Leer (HLL) flux is applied to discrete the inviscid term of the variable density incompressible Navier-Stokes equations. The viscous term is discretized by the direct DG (DDG) method, the construction of which was initially inspired by the weak solution of a scalar diffusion equation. In addition, in order to eliminate the spurious oscillations around sharp density gradients, a local slope limiting operator is also applied during the highly stratified flow simulations. The convergence property and performance of the present high-order DDG method are well demonstrated by several benchmark and challenging numerical test cases. Due to its advantages of simplicity and robustness in implementation, the present method offers an effective approach for simulating the variable density incompressible flows.

AMS subject classifications: 35Lxx, 65Mxx, 65Nxx

Key words: Variable density incompressible flows, direct discontinuous Galerkin method, artificial compressibility, high-order accuracy.

1 Introduction

The incompressible flows with variable density exist widely in the field of fluid dynamics, e.g. in the problems of ground water flows, highly stratified flows and density-driven flows in porus media [1]. However, very few researches have been devoted to the numerical simulation of this problem although the existence and uniqueness of solutions to its

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governing equations, i.e. the variable density incompressible Navier-Stokes equations, has already been well analyzed in literature [2].

Actually, the numerical methods encounter difficulties in solving both constant and variable density incompressible Navier-Stokes equations. The main reason lies in the lack of time-derivative term of density in the continuity equation and therefore the explicit temporal integration scheme cannot be employed directly to update the density and pressure fields. Besides, since the inviscid form of the governing equations is not hyperbolic, the Riemann problem based technique cannot be used in the construction of high-order numerical methods including the finite volume (FV) method and the discontinuous Galerkin (DG) method, etc. In order to overcome the above difficulties, one kind of approach is the projection method which was initially developed by Chorin [3] and Teman [4] for solving the incompressible Navier-Stokes equations, and then extended by Guermond and Quartapelle [5] and Li et al. [6] for solving the variable density incompressible Navier-Stokes equations. Another kind of approach is the artificial compressibility method which was firstly proposed by Chorin [7]. Its main principle is to add an artificial compressibility term (pseudo-time derivative of pressure) into the continuity equation to reformulate a hyperbolic system of governing equations. The artificial compressibility term is supposed to vanish as the steady state is reached. Based on the artificial compressibility concept, Bassi et al. [8, 9] developed an effective approach for solving the incompressible flows in the context of DG formulation. In their work, the artificial compressibility term is employed only in the flux construction procedure with the sole purpose of recovering the hyperbolic nature of the governing equations at cell interfaces. It guarantees that the consistency of the resulting DG discretization is unaffected by the amount of artificial compressibility. Later, this kind of artificial compressibility inviscid flux formulation has also been successfully extended to the variable density incompressible flow simulations [10].

Inspired by the aforementioned studies, Zhang et al. [11–13] recently developed a novel high-order DG method for solving the incompressible Navier-Stokes equations on arbitrary grids. In order to improve the computational efficiency, a simplified artificial compressibility flux formulation is carefully designed by employing an exact Riemann solver to discretize the linear Stokes operator and a local Lax-Friedrichs (LLF) flux to discretize the nonlinear convective term. The direct DG (DDG) method, which was initially proposed to solve the scalar diffusion equation by Liu and Yan [14,15] and later proved to be an attractive approach for solving the compressible Navier-Stokes equations by Cheng et al. [16–18], is used to discretize the viscous term. Due to its simplicity in implementation, in this work, we will extend this high-order DDG method to simulate the variable density incompressible flows. Besides, in order to eliminate the possible spurious oscillations occurred in the vicinity of sharp density gradient regions, a local slope limiting operator is employed during the numerical simulation of highly stratified flow problems. Several benchmark and challenging problems are tested to demonstrate the convergence property and performance of the high-order DDG method. The numerical results show that the present method is well capable of simulating the variable density incompressible