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# A New Directional Algebraic Fast Multipole Method Based Iterative Solver for the Lippmann-Schwinger Equation Accelerated with HODLR Preconditioner

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**Abstract.** We present a fast iterative solver for scattering problems in 2D, where a penetrable object with compact support is considered. By representing the scattered field as a volume potential in terms of the Green's function, we arrive at the Lippmann-Schwinger equation in integral form, which is then discretized using an appropriate quadrature technique. The discretized linear system is then solved using an iterative solver accelerated by Directional Algebraic Fast Multipole Method (DAFMM). The DAFMM presented here relies on the directional admissibility condition of the 2D Helmholtz kernel [1], and the construction of low-rank factorizations of the appropriate low-rank matrix sub-blocks is based on our new Nested Cross Approximation (NCA) [2]. The advantage of the NCA described in [2] is that the search space of so-called far-field pivots is smaller than that of the existing NCAs [3,4]. Another significant contribution of this work is the use of HODLR based direct solver [5] as a preconditioner to further accelerate the iterative solver. In one of our numerical experiments, the iterative solver does not converge without a preconditioner. We show that the HODLR preconditioner is capable of solving problems that the iterative solver can not. Another noteworthy contribution of this article is that we perform a comparative study of the HODLR based fast direct solver, DAFMM based fast iterative solver, and HODLR preconditioned DAFMM based fast iterative solver for the discretized Lippmann-Schwinger problem. To the best of our knowledge, this work is one of the first to provide a systematic study and comparison of these different solvers for various problem sizes and contrast functions. In the spirit of reproducible computational science, the implementation of the algorithms developed in this article is made available at `https://github.com/vaishna77/Lippmann_Schwinger_Solver`.

**AMS subject classifications**: 31A10, 35J05, 35J08, 65F55, 65R10, 65R20

**Key words**: Directional Algebraic Fast Multipole Method, Lippmann-Schwinger equation, low-rank matrix, Helmholtz kernel, Nested Cross Approximation, HODLR direct solver, Preconditioner.

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# 1 Introduction

This article focuses on developing a fast iterative solver for scattering problems in 2D. Consider a penetrable object with an electric susceptibility (or *contrast function*) of $q(x)$. Assume $q(x)$ to have compact support in a domain $\Omega$. Let $u^{inc}(x)$ be the incident field and $u^{scat}(x)$ be the unknown scattered field. Let $\kappa$ be the wavenumber of the incident field. The total field $u$, which is the sum of incident and scattered fields, follows the time-harmonic Helmholtz equation

$$\nabla^2 u(x)+\kappa^2(1+q(x))u(x)=0, \quad x\in\mathbb{R}^2. \tag{1.1}$$

The incident field satisfies the homogeneous Helmholtz equation

$$\nabla^2 u^{inc}(x)+\kappa^2 u^{inc}(x)=0, \quad x\in\Omega. \tag{1.2}$$

It follows from Eq. (1.1) and Eq. (1.2) that $u^{scat}(x)$ satisfies

$$\nabla^2 u^{scat}(x)+\kappa^2(1+q(x))u^{scat}(x)=-\kappa^2 q(x)u^{inc}(x), \quad x\in\Omega. \tag{1.3}$$

To ensure the scattered field propagates to infinity without any spurious resonances, we enforce the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u}{\partial r}^{scat}-i\kappa u^{scat}\right)=0, \quad \text{where } r=||x||. \tag{1.4}$$

There exist many techniques to solve the scattered field. A few of them worth mentioning are: constructing a variational form, discretizing the differential operator, reformulating it as a volume integral equation. We use the volume integral equation technique as described in [6], where the scattered field is expressed as a volume potential

$$u^{scat}(x)=V[\psi](x)=\int_\Omega G_\kappa(x,y)\psi(y)dy, \quad x\in\Omega, \tag{1.5}$$

where

$$G_\kappa(x,y)=\frac{i}{4}H_0^{(1)}(\kappa||x-y||) \tag{1.6}$$

is the Green's function of Helmholtz equation in 2D. Using Eq. (1.3) and Eq. (1.5), we obtain the Lippmann-Schwinger equation

$$\psi(x)+\kappa^2 q(x)V[\psi](x)=f(x), \quad x\in\Omega, \tag{1.7}$$

where $f(x)=-\kappa^2 q(x)u^{inc}(x)$. The task is to numerically solve for $\psi(x)$ and then obtain $u^{scat}(x)$.

The present article discusses a fast iterative solver for the Lippmann-Schwinger equation, developed on an adaptive grid. Iterative solvers rely on matrix-vector products,