

# Poisson Integrators Based on Splitting Method for Poisson Systems

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Received 18 May 2022; Accepted (in revised version) 19 September 2022

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**Abstract.** We propose Poisson integrators for the numerical integration of separable Poisson systems. We analyze three situations in which Poisson systems are separated in three ways and Poisson integrators can be constructed by using the splitting method. Numerical results show that the Poisson integrators outperform the higher order non-Poisson integrators in terms of long-term energy conservation and computational cost. The Poisson integrators are also shown to be more efficient than the canonicalized symplectic methods of the same order.

**AMS subject classifications:** 65P10

**Key words:** Poisson systems, Poisson integrators, splitting method, energy conservation.

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## 1 Introduction

In this paper we propose Poisson integrators for the numerical integration of Poisson systems [16, 20, 24] with separable Hamiltonian. Poisson systems have Poisson structures which are preserved by Poisson integrators. There is no universal approach to constructing Poisson integrators for all Poisson systems. However, by using the splitting method [6, 29, 37], one can construct Poisson integrators for separable Poisson systems. We identify three situations in which Poisson systems are separated in three ways and Poisson integrators can be constructed.

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Poisson systems are the generalized canonical Hamiltonian systems where the constant matrix  $J^{-1}$  is replaced by a variable-dependent matrix  $R(Z)$ . They appear in a variety of scientific disciplines, such as the celestial mechanics, quantum mechanics, plasma physics and fluid dynamics. The well-known Poisson systems are the Euler equations for the rigid body [39], the nonlinear Schrödinger equations [13, 38], the charged particle system [21, 23, 43, 44], the gyrocenter system [34, 42, 46], the Maxwell-Vlasov equations [22, 32], the ideal MHD equations [33] and the isentropic compressible fluids. The phase flow of the Poisson system is usually very difficult to obtain. Thus, it is critical to construct accurate and efficient numerical integrators with long-term conservation property and stability. Poisson integrators, like symplectic methods [9, 14, 17, 20, 38] for canonical Hamiltonian systems [3, 4, 15, 36], exhibit advantageous structure-preserving properties [9, 14, 17]. Meanwhile, Poisson integrators have the property of long-term energy conservation. Therefore, we will formulate the construction of Poisson integrators for Poisson systems.

Many researchers have paid attention to investigating Poisson integrators for Poisson systems, including the theoretical results on the construction of the integrators by using the generating function [18, 19], splitting method [27], the Lie algebra [10, 11, 31] and other techniques [5, 40, 47] and the application of the integrators to the Schrödinger equation [13], the rigid body problem [39] and the charged particle system [21]. Ge and Marsden proposed the Lie-Poisson integrator that exactly preserves the Lie-Poisson structure based on the generating function which is derived as an approximate solution of Hamiltonian-Jacobi equation [19]. Ge developed the Lie-Poisson integrator by applying the equivalence theorem of Feng-Ge to the first kind of generating function [18]. Explicit Lie-Poisson integrators using the splitting method were developed for Lie-Poisson systems [27]. Channel and Scovel reformulate the integrator of Ge and Marsden in terms of algebra variables and implement it to arbitrary high order for regular quadratic Lie algebra [10]. Modin and Viviani proposed a class of isospectral symplectic Runge-Kutta methods for Lie-Poisson systems on a reductive Lie algebra, which preserve the Lie-Poisson structure [31]. The Schrödinger midpoint method was developed and applied to general Lie-Poisson systems as a special case of isospectral midpoint method [11]. Some Lie-Poisson integrators were constructed for the Poisson system with constant matrix  $R(Z)$  [5, 40, 47]. Efficient symplectic integration schemes were also constructed for Lie-Poisson systems [30]. For the application of Poisson integrators, Faou and Lubich derived a symmetric Poisson integrator using the variational splitting technique based on the discovery that the Hamiltonian reduction of the Schrödinger equation to the Gaussian wavepacket manifold inherits a Poisson structure [13]. Touma and Wisdom derived a symplectic integrator for a free rigid body and incorporated this integrator in the  $n$ -body integrator [41] to provide a Lie-Poisson integrator for the one or more rigid bodies dynamics [39]. Recently, the splitting technique has been applied to construct Poisson integrators for Poisson systems. Non-canonical Hamiltonian systems are special Poisson systems where the matrix  $R(Z)$  is invertible. Zhu et al. investigated the particular situations that the explicit K-symplectic schemes can be constructed for non-canonical