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The Corrected Finite Volume Element Methods for Diffusion Equations Satisfying Discrete Extremum Principle

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Abstract. In this paper, we correct the finite volume element methods for diffusion equations on general triangular and quadrilateral meshes. First, we decompose the numerical fluxes of original schemes into two parts, i.e., the principal part with a two-point flux structure and the defective part. And then with the help of local extremums, we transform the original numerical fluxes into nonlinear numerical fluxes, which can be expressed as a nonlinear combination of two-point fluxes. It is proved that the corrected schemes satisfy the discrete strong extremum principle without restrictions on the diffusion coefficient and meshes. Numerical results indicate that the corrected schemes not only satisfy the discrete strong extremum principle but also preserve the convergence order of the original finite volume element methods.

AMS subject classifications: 65N08

Key words: Diffusion equations, finite volume element, flux-correct, maximum principle.

1 Introduction

In this article, we study the diffusion equations with Dirichlet boundary conditions:

$$\begin{cases} -\nabla\cdot(\kappa(x,y)\nabla u)=f, & \text{in } \Omega, \\ u=g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is an open bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, the symmetric diffusion coefficient $\kappa(x,y)=\left(\kappa_{ij}(x,y)\right)_{i,j=1,2}$ satisfies the uniformly elliptic condition on Ω, i.e., there exist constants $0<c_1<c_2$, such that

$$c_1|\xi|^2\leq(\kappa(x,y)\xi,\xi)\leq c_2|\xi|^2,\quad \forall\xi\in\mathbb{R}^2. \tag{1.2}$$

Furthermore, the source term $f\in L^2(\Omega)$, and g is a boundary function defined on $\partial\Omega$.

Diffusion operator appears in many physical models such as reservoir modeling, and energy transport in inertial confinement fusion. In the process of numerical simulation, we not only require the schemes to have well convergence, but also satisfy some physical properties, such as local conservation and extreme principle. The strong extremum principle is equivalent to the second law of thermodynamics. Numerical simulation of some complex problems strongly depends on whether the scheme satisfies the extremum principle, for example, the convection-dominated convection diffusion problems [1] and the phase field problems [2–4]. However, the influence of mesh and diffusion coefficient may lead to non-physical oscillation, and the numerical solution violates the discrete extremum principle. Many researchers focus on developing numerical schemes that satisfy the discrete maximum principle (DMP). The finite volume methods have local conservation property, and are widely used in numerical simulation for partial differential equations [5]. The finite volume methods usually include two types: cell-centered type and vertex-centered type.

Many scholars have made great efforts in the development of cell-centered finite volume schemes which satisfy the extremum principle [6,7]. Nordbotten et al. point out that linear nine-point schemes cannot both unconditionally satisfy DMP and have second-order accuracy [8]. Hence many nonlinear techniques are applied to the development of finite volume schemes. In [9], Droniou et al. introduce the local maximum principle (LMP) structure that ensures the discrete local extremum principle. The coefficient matrix corresponding to the LMP structure is an M-matrix, which gives a sufficient condition of DMP. Under a rigorous restriction on simplex meshes, which is equivalent to non-obtuse angle conditions in two-dimensional space, Bertolazzi et al. propose a second-order nonlinear finite volume scheme satisfying DMP for steady convection-diffusion problems in [10]. To obtain an extremum-preserving scheme on general meshes, auxiliary unknowns at harmonic averaging points are used in works [11–14]. However, the robustness of the above schemes is affected by the distribution relationship of interpolation nodes and auxiliary nodes. In fact, for those schemes in [11–14], there are some restrictions on the positions of harmonic averaging points, which lead to certain constraints on the geometries of mesh-cells and diffusion coefficients. In [15], Sheng and Yuan propose an extremum-principle-preserving finite volume scheme, which can be applied to arbitrary polygonal meshes, but still has to impose some restrictions on diffusion coefficients to ensure accurate and fast resolution of discontinuity. Based on the similar idea, a vertex-centered finite volume scheme preserving DMP is proposed in [16]. Recently, Yuan proposes a new nonlinear correction technique in [17] and applies it to a