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Bjorken Flow Revisited: Analytic and Numerical Solutions in Flat Space-Time Coordinates

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Abstract. In this work we provide analytic and numerical solutions for the Bjorken flow, a standard benchmark in relativistic hydrodynamics providing a simple model for the bulk evolution of matter created in collisions between heavy nuclei.

We consider relativistic gases of both massive and massless particles, working in a (2+1) and (3+1) Minkowski space-time coordinate system. The numerical results from a recently developed lattice kinetic scheme show excellent agreement with the analytic solutions.

AMS subject classifications: 76Y05

Key words: Relativistic hydrodynamics, heavy-ion collisions, Bjorken flow, lattice kinetic solvers.

1 Introduction

In recent years, experimental data from the Relativistic Heavy-Ion Collider (RHIC) and the Large Hadron Collider (LHC) [1–5] has provided the first clear observation of the Quark-Gluon Plasma (QGP), a deconfined phase of matter where quarks and gluons are effectively free beyond the nucleonic volume [6–8].

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Remarkably, the earliest stages following the heavy-ion collisions present collective behaviors that can be described by the laws of fluid dynamics [9], and indeed these results have significantly boosted the interest in the study of viscous relativistic fluid dynamics [10], both at the level of theoretical formulations as well as in the development of reliable numerical simulation methods.

Most of the numerical methods are based on Israel-Stewart theory [11] or more recent second-order causal formalism [12], with a few example represented by MUSIC [13], vSHASTA [14], ECHO-QGP [15], and several more (see e.g. [16,17] and citations therein). Besides, mesoscopic approaches are often employed in order to study relativistic hydrodynamic systems, including lattice kinetic schemes [18], as well as Monte Carlo based methods [19,20].

The development of new numerical tools for the study of relativistic fluids is an active field of research. However evaluating and comparing the accuracy, stability and performance of the available solvers represent a challenging task, due to the fact that numerical benchmarks with an analytic solution are rare and available only in very idealized situations.

Two of the most commonly used benchmarks are the Riemann problem [21,22], for which an analytic solution is available only in the inviscid limit (see [23–27] for numerical results in the viscous regime), and the Bjorken flow [28] (as well as its generalization given by the Gubser flow [29]).

The Bjorken flow represents the simplest numerical setup in the QGP context. Under the assumption that all particles get their velocity at the initial collision, the Bjorken flow describes the boost-invariant longitudinal (*i.e.* along the heavy-ion beams) expansion of the QGP. Because of its formulation, this flow is naturally described recurring to the Milne coordinate system, where the macroscopic velocity results at rest. The formulation of the flow in a static Minkowski space-time is on one hand more complex, but potentially useful in the validation of codes working in Cartesian laboratory frame coordinates.

In this work, we present the analytic solution for the Bjorken flow, considering a inviscid fluid consisting of both massive and massless particles, working in a $(2+1)$ and $(3+1)$ Minkowski space-time coordinate system.

We provide details for the implementation of the benchmark using the Relativistic Lattice Boltzmann Method (RLBM) [18], a class of numerical models providing a computational efficient approach for the solution of the relativistic Boltzmann equation in the relaxation time approximation. Since lattice kinetic models rely on a mesoscopic description of the dynamics viscous effects are naturally included, with relativistic invariance and causality preserved by construction (this at variance with respect to aforementioned hydrodynamic models).

Numerical results are shown to be in excellent agreement with the analytic solutions.

This paper is organized as follows: in Section 2 we provide a brief introduction on the notation and the relevant equations for the kinetic description of a relativistic fluid. In Section 3 we provide analytic solutions for the Bjorken flow in a Minkowski space-time for fluids of both massive and massless particles. In Section 4 we report the implementa-