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A One-Dimensional Second-Order Cell-Centered Lagrangian Scheme Satisfying the Entropy Condition

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Abstract. The numerical solutions of gas dynamics equations have to be consistent with the second law of thermodynamics, which is termed entropy condition. However, most cell-centered Lagrangian (CL) schemes do not satisfy the entropy condition. Until 2020, for one-dimensional gas dynamics equations, the first-order CL scheme with the hybridized flux developed by combining the acoustic approximate (AA) flux and the entropy conservative (EC) flux developed by Maire et al. was used. This hybridized CL scheme satisfies the entropy condition; however, it is under-entropic in the part zones of rarefaction waves. Moreover, the EC flux may result in nonphysical numerical oscillations in simulating strong rarefaction waves. Another disadvantage of this scheme is that it is of only first-order accuracy. In this paper, we firstly construct a modified entropy conservative (MEC) flux which can damp effectively numerical oscillations in simulating strong rarefaction waves. Then we design a new hybridized CL scheme satisfying the entropy condition for one-dimensional complex flows. This new hybridized CL scheme is a combination of the AA flux and the MEC flux.

In order to prevent the specific entropy of the hybridized CL scheme from being under-entropic, we propose using the third-order TVD-type Runge-Kutta time discretization method. Based on the new hybridized flux, we develop the second-order CL scheme that satisfies the entropy condition.

Finally, the characteristics of our new CL scheme using the improved hybridized flux are demonstrated through several numerical examples.

AMS subject classifications: 65M08

Key words: Cell-centered Lagrangian scheme, entropy conditions, modified entropy conservative flux, second-order scheme.

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1 Introduction

This study aims to first analyze the entropy condition of some widely-used first-order cell-centered Lagrangian (CL) schemes and then develop new hybridized CL schemes based on the existing hybridized CL scheme for one-dimensional equations of gas dynamics developed by Maire et al. [1] in 2020.

The equations of gas dynamics are a non-linear and hyperbolic system that can generate shock waves at a finite time even under initial smoothing conditions [2]. To seek physically relevant solutions to the equations, we need the equations' numerical solutions to satisfy the entropy condition.

This results in the solutions of gas dynamics equations being forced to be consistent with the second law of thermodynamics where rarefaction waves are a reversible process with the conserved entropy, whereas shock waves are an irreversible process with increased entropy to ensure the conversion of kinetic energy to internal energy.

There are two types of Lagrangian schemes: a staggered Lagrangian (SL) scheme and a CL scheme. The SL scheme was first proposed by J. Von Neumann and R. D. Richtmyer [3], which was later developed by Caramana et al. [4] in 1998. In the SL scheme, the kinematic variables are defined at the mesh vertices, while the thermodynamic variables at the cell center, the specific internal energy equation is discretized, and the artificial viscosity is added to the gas-dynamics equations to deal with shock waves. Moreover, only in zones compressed, not in zones expanded, the artificial viscosity is used, which ensures the SL scheme satisfies the entropy condition: the entropy of the SL scheme is conserved across rarefaction waves, and is increased across shock waves. A CL scheme in one dimension was firstly designed by Godunov [5,6]. In the CL scheme, all variables are defined at the cell centers, the total energy equation is discretized, the mass, momentum and total energy are conserved, and the Riemann solvers are used to compute the numerical fluxes at the cell interfaces. Moreover, in simulating non-smooth flows, entropy was increased, which satisfies the entropy condition; but for smooth flows, the entropy was also increased, which violates the second law of thermodynamics regarding smooth flows. During the eighties, people had extended this method to two dimensions, see [7,8], but these schemes in two dimensions were not consistent with the second law of thermodynamics. This incompatible problem had been partly solved by Després and Mazeran in [9] and Maire et al. in [10]. The schemes developed by them satisfy the entropy increase principle in simulating shock waves, but violate the entropy conservation principle in simulating rarefaction waves, which is the most important flaw of these CL schemes.

This flaw was addressed by Maire et al. [1] in 2020. For one-dimensional gas dynamics, Maire et al. first developed an entropy conservation (EC) numerical flux by extending Tadmor's work [11] from the Eulerian framework to the Lagrangian framework. They then built a hybrid entropy-consistent numerical flux (HAE), which is the combination of the acoustic approximation flux and the EC flux.

This hybridized flux is degenerated to the EC flux in the expanded zones, while the