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A Posteriori Error Estimate of Weak Galerkin FEM for Stokes Problem Using Auxiliary Subspace Techniques

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Abstract. Based on the auxiliary subspace techniques, *a posteriori* error estimator of nonconforming weak Galerkin finite element method (WGFEM) for Stokes problem in two and three dimensions is presented. Without saturation assumption, we prove that the WGFEM approximation error is bounded by the error estimator up to an oscillation term. The computational cost of the approximation and the error problems is considered in terms of size and sparsity of the system matrix. To reduce the computational cost of the error problem, an equivalent error problem is constructed by using diagonalization techniques, which needs to solve only two diagonal linear algebraic systems corresponding to the degree of freedom ($d.o.f$) to get the error estimator. Numerical experiments are provided to demonstrate the effectiveness and robustness of the *a posteriori* error estimator.

AMS subject classifications: 65N15, 65N30, 76D07

Key words: Auxiliary subspace techniques, diagonalization techniques, weak Galerkin, *A posteriori* error estimate, Stokes problem.

1 Introduction

This paper concerns a hierarchical basis *a posteriori* estimation of error in WGFEM approximation [18,19,21–23] for Stokes problem of the forms:

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$$-\nu\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u}=0 \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}=\boldsymbol{g} \quad \text{on } \Gamma. \tag{1.3}$$

Here, $\Omega\subset\mathbb{R}^d(d=2,3)$ is a bounded polygonal or polyhedral domain with boundary Γ, $\boldsymbol{u}$ is a vector velocity field and p is the pressure. The problem (1.1)-(1.3) has a unique solution in the sense that p is determined only up to an additive constant. $\boldsymbol{f}$ and $\boldsymbol{g}$ are given Lebesgue square integrable functions on Ω and Γ, respectively. ν is the dynamic viscosity.

The hierarchical basis *a posteriori* estimator is one of the most successful estimators, whose origins can be traced back to [26,27]. For the reliability and efficiency with the energy error, the main tool in classical proof [3, 11, 17] is the saturation assumption, which has been shown to be superfluous in [8, 14]. To remove the saturation assumption, an approach proposed by Araya et al. is to construct some suitable bubble functions satisfying a discrete Babuška-Brezzi condition to prove the reliability and efficiency, which has been applied in advection-diffusion-reaction problem [5], generalized Stokes problem [4], and Navier-Stokes equations [6, 7]. Using some special bubble functions as auxiliary subspace, Hakula et al. give a different approach to prove the reliability and efficiency without the saturation assumption for the second order elliptic problem [12] and elliptic eigenvalue problem [9]. The authors of this paper have extended the auxiliary subspace techniques in [12] to Taylor-Hood element for Stokes problem [24]. Another different approach in how to overcome the saturation assumption by Nochetto et al. can be found in [1, 14].

One of the main contributions of this work is to extend the auxiliary subspace techniques in [12] to WGFEM for Stokes problem combining with a Helmholtz-like decomposition proposed in [25] in two and three dimensions. The approximation problem (2.9) is discretized by WGFEM with a variational formulation which is different from that shown in (2.1) satisfied by true solution and used to solve error problem (2.17). The difference of variational formulations between the approximation and the original problem makes it more difficult to prove the reliability and efficiency than in [24], which uses conforming finite element method (FEM) to construct the approximation problem such that the same variational formulation is appropriate for the two problems. One of the difficulties is that the error between true and numerical solutions can not be written in any of the aforementioned variational formulations. As a result, we can not obtain the reliability and efficiency by using the techniques in [24]. Therefore, we will use a novel approach with the help of a Helmholtz-like decomposition previously developed in [25] and a suitable quasi-interpolant previously developed in [24] to show the reliability and efficiency up to an oscillation term shown in Theorem 3.1.

The other contribution of this paper is to reduce the computational cost of the error problem by using a diagonalization technique. To begin with, we analyse and compare