Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2022-0284

Vol. **33**, No. 4, pp. 1090-1105
April 2023

An Efficient Neural-Network and Finite-Difference Hybrid Method for Elliptic Interface Problems with Applications

Wei-Fan Hu[1,4], Te-Sheng Lin[2,4,*], Yu-Hau Tseng[3] and Ming-Chih Lai[2]

[1] *Department of Mathematics, National Central University, Taoyuan 32001, Taiwan.*
[2] *Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu 30010, Taiwan.*
[3] *Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 81148, Taiwan.*
[4] *National Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan.*

Received 6 November 2022; Accepted (in revised version) 4 March 2023

Abstract. A new and efficient neural-network and finite-difference hybrid method is developed for solving Poisson equation in a regular domain with jump discontinuities on embedded irregular interfaces. Since the solution has low regularity across the interface, when applying finite difference discretization to this problem, an additional treatment accounting for the jump discontinuities must be employed. Here, we aim to elevate such an extra effort to ease our implementation by machine learning methodology. The key idea is to decompose the solution into singular and regular parts. The neural network learning machinery incorporating the given jump conditions finds the singular solution, while the standard five-point Laplacian discretization is used to obtain the regular solution with associated boundary conditions. Regardless of the interface geometry, these two tasks only require supervised learning for function approximation and a fast direct solver for Poisson equation, making the hybrid method easy to implement and efficient. The two- and three-dimensional numerical results show that the present hybrid method preserves second-order accuracy for the solution and its derivatives, and it is comparable with the traditional immersed interface method in the literature. As an application, we solve the Stokes equations with singular forces to demonstrate the robustness of the present method.

AMS subject classifications: 65N06, 65N99, 35J25

Key words: Neural networks, sharp interface method, fast direct solver, elliptic interface problem, Stokes equations.

*Corresponding author. *Email addresses:* `wfhu@math.ncu.edu.tw` (W.-F. Hu), `tslin@math.nctu.edu.tw` (T.-S. Lin), yhtseng@nuk.edu.tw (Y.-H. Tseng), `mclai@math.nctu.edu.tw` (M.-C. Lai)

1 Introduction

In this paper, we aim to solve a d-dimensional ($d=2$ or 3) elliptic interface problem defined in a regular domain $\Omega\subset\mathbb{R}^d$, which is separated by an embedded interface Γ such that the subdomains inside and outside the interface are denoted by Ω^- and Ω^+, respectively. Along the interface Γ, there exists jump discontinuities that the solution must be satisfied. With the associated boundary condition, the problem takes the form

$$\Delta u(\mathbf{x})=f(\mathbf{x}),\quad \mathbf{x}\in\Omega^-\cup\Omega^+, \tag{1.1}$$

$$[\![u(\mathbf{x})]\!]=\gamma(\mathbf{x}),\quad [\![\partial_n u(\mathbf{x})]\!]=\rho(\mathbf{x}),\quad \mathbf{x}\in\Gamma, \tag{1.2}$$

$$u(\mathbf{x})=u_b(\mathbf{x}),\quad \mathbf{x}\in\partial\Omega. \tag{1.3}$$

Here, the jump $[\![\cdot]\!]$ indicates the quantity approaching from Ω^+ side minus the one from Ω^- side; the shorthand $\partial_n u$ represents the normal derivative $\nabla u\cdot\mathbf{n}$ in which $\mathbf{n}$ is the normal vector pointing from Ω^- to Ω^+. Notice that, here the underlying differential equation is subject to the Dirichlet-type boundary condition for illustration purpose, while other types of boundary condition (Neumann or Robin) will not change the main ingredients presented here. Since the Poisson equation is considered in Eq. (1.1), we simply call the above problem as the Poisson interface problem hereafter.

As seen from Eqs. (1.1)-(1.3), the solution and its partial derivatives have jumps across the interface. So, when applying the finite difference discretization to this problem, an additional treatment accounting for those jump discontinuities must be employed at the grid points near the interface. Over the past few decades, different discretization methodologies have been successfully developed to capture those jump conditions sharply or to improve the overall numerical accuracy, such as the immersed interface method (IIM) [12, 13,16,18], ghost fluid method (GFM) [6,22], Voronoi interface method [7], to name a few. Different approaches for solving interface problems such as the immersed finite element method (IFEM) [8,10] or other methods can be found in [20] and the references therein.

On the other hand, much attention has recently been paid to applying deep neural networks (DNNs) to solve elliptic interface problems, rather than using traditional numerical methods to solve such problems. Despite the success of the two mainstream deep learning approaches (Physics-Informed Neural Networks (PINNs) [25,26] and the deep Ritz method [5]) in solving partial differential equations with smooth solutions, learning methods based on these two frameworks for solving elliptic interface problems with jump discontinuities remain to be improved. The main and intrinsic difficulty may be attributed to the fact that the usual activation functions used in DNNs are generally smooth; thus, DNN function approximators seem to be incapable of representing discontinuous functions. To approximate such discontinuous solutions (or functions) and tackle the elliptic interface problems, multiple independent networks need to be established and linked with each other by imposing the jump conditions, see, e.g., piecewise DNNs [11], interfaced neural networks [27], and deep unfitted Nitsche method [9]. The resulting prediction errors in their test examples reach the magnitude $\mathcal{O}(10^{-3})$ to $\mathcal{O}(10^{-4})$ in relative L^2 norm. Moreover, training these DNN models comes at the cost of having to