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New Superconvergent Structures with Optional Superconvergent Points for the Finite Volume Element Method

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Abstract. New superconvergent structures are proposed and analyzed for the finite volume element (FVE) method over tensorial meshes in general dimension d (for $d\geq 2$); we call these *orthogonal superconvergent structures*. In this framework, one has the freedom to choose the superconvergent points of tensorial k-order FVE schemes (for $k\geq 3$). This flexibility contrasts with the superconvergent points (such as Gauss points and Lobatto points) for current FE schemes and FVE schemes, which are fixed. The orthogonality condition and the modified M-decomposition (MMD) technique that are developed over tensorial meshes help in the construction of proper superclose functions for the FVE solutions and in ensuring the new superconvergence properties of the FVE schemes. Numerical experiments are provided to validate our theoretical results.

AMS subject classifications: 65N12, 65N08, 65N30

Key words: Superconvergence, finite volume, orthogonality condition, tensorial mesh, rectangular mesh.

1 Introduction

Superconvergence is the phenomenon that the numerical solution (or the postprocessed solution) has higher convergence rate at certain points or with certain metric than generally expected. It is an efficient way to improve the efficiency and accuracy for numerical methods such as the finite element method (FEM) [1,3,16,19,21,26,28,32,34–36,46] and the finite volume element method (FVEM) [5,7,17,31,48] etc. In the past decades, many progresses have been made on the natural superconvergence [2,6,8,14,27,39,50], the global

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superconvergence [2, 13, 22, 39] and the post-processed superconvergence [4, 43, 47, 51], which are the three main aspects of the superconvergence research.

The finite volume element method is one of the main methods for spatial discretizations. Due to the local conservation property, the FVEM has been widely applied in scientific and engineering computations [5,9,12,13,23,24,33,37]. It is well known that the trial and test function spaces are chosen differently, which makes the theoretical analysis of the finite volume element method complicated and difficult. Meanwhile, this setting of different spaces provides us some flexibility in constructing FVE schemes. Readers are referred to [11,12,18,25,42,48,49] for stability and H^1 estimates, [10,15,20,29,31,38,40,45] for L^2 estimates, and [6, 7, 30, 39, 41] for superconvergence of the finite volume element method. To the authors' knowledge, almost all existing superconvergence results are associated to fixed superconvergent points (natural superconvergence), such as the famous Gauss-Lobatto structure for the FEM/FVEM.

This paper concerns with a class of new superconvergent structures of the FVEM over tensorial meshes in general d-dimension ($d \geq 2$). We find that these superconvergent structures cover the Gauss-Lobatto structure and include more superconvergent structures developed by some new FVE schemes. From the construction of tensorial r-order FVEM, in each direction, one has $(l-1)$-degrees ($r=2l$ or $r=2l-1$, $l \geq 1$) of freedom to choose superconvergent points of gradient value, and has $(l-1)$-degrees ($r=2l$ or $r=2l+1$, $l \geq 1$) of freedom to choose superconvergent points of function value. Furthermore, for $r \geq 3$, one can choose r symmetric gradient superconvergent points in each direction for odd r, and $(r+1)$ symmetric function value superconvergent points in each direction for even r. Moreover, the superconvergent points can be chosen differently in different directions. These provide us some interesting new schemes, such as the tensorial 4-order FVE scheme which has uniform superconvergent points of function value (see Example 5.1), and the tensorial 5/6-order FVE scheme which holds superconvergence properties of the function values and gradient values simultaneously at some points (see Example 5.2).

Theoretically, the proof of natural superconvergence mainly consists of two aspects: 1) constructing a superclose function u_I such that $|\nabla(u-u_I)| = \mathcal{O}(h^{r+1})$ at gradient value superconvergent points and/or $|u-u_I| = \mathcal{O}(h^{r+2})$ at function value superconvergent points; 2) proving the global superconvergence results that $\|u_h-u_I\|_1 = \mathcal{O}(h^{r+1})$ and/or $\|u_h-u_I\|_0 = \mathcal{O}(h^{r+2})$. By proposing the modified M-decomposition (MMD) over tensorial meshes, we construct a proper superclose function for the FVE scheme. Combining with the proposed tensorial orthogonality condition with the MMD, we prove the superconvergence results: when a FVE scheme satisfies the tensorial r-$(r-1)$-order orthogonality condition, it has the superconvergence of gradient value; and when it satisfies the tensorial r-r-order orthogonality condition, it has the superconvergence of function value. We would like to emphasize that, the main role of the MMD is to construct the superclose function in the proof, and the effect of the orthogonality condition on the superconvergence property is more essential.

For simplicity, the superconvergence results are presented for the model problem (2.1), all results obtained here are valid for convection-diffusion-reaction problems over