Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2022-0187

Vol. **34**, No. 5, pp. 1215-1246
November 2023

Operator Splitting and Local Time-Stepping Methods for Transport Problems in Fractured Porous Media

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Received 10 July 2022; Accepted (in revised version) 29 November 2022

Abstract. This paper is concerned with efficient numerical methods for the advection-diffusion equation in a heterogeneous porous medium containing fractures. A dimensionally reduced fracture model is considered, in which the fracture is represented as an interface between subdomains and is assumed to have larger permeability than the surrounding area. We develop three global-in-time domain decomposition methods coupled with operator splitting for the reduced fracture model, where the advection and the diffusion are treated separately by different numerical schemes and with different time steps. Importantly, smaller time steps can be used in the fracture-interface than in the subdomains. The first two methods are based on the physical transmission conditions, while the third one is based on the optimized Schwarz waveform relaxation approach with Ventcel-Robin transmission conditions. A discrete space-time interface system is formulated for each method and is solved iteratively and globally in time. Numerical results for two-dimensional problems with various Péclet numbers and different types of fracture are presented to illustrate and compare the convergence and accuracy in time of the proposed methods with local time stepping.

AMS subject classifications: 65M55, 65N30, 76S05, 35K20

Key words: Domain decomposition, reduced fracture model, operator splitting, nonconforming time grids, time-dependent Steklov-Poincaré operator, optimized Schwarz waveform relaxation.

1 Introduction

The need for accurate numerical simulations of fluid flow and transport problems in porous media is well-recognized because of their application in various fields such as

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subsurface hydrology, geophysics, and reservoir geomechanics. However, it is often challenging to model such problems due to the domain of calculation being a combination of many sub-regions with different processes, which leads to the presence of multiple spatial and temporal scales and drastically different physical properties. In particular, this is the case for a domain where there exist fractures and faults. In such a case, the time scales in the fractures may be significantly smaller or bigger than in the subdomains depending on whether the permeability in the fractures is much higher or lower than in the surrounding rock matrix. Additionally, the width of the fractures is much smaller than the size of the domain of calculation and any reasonable parameter of spatial discretization. One effective way to deal with this situation is to treat the fractures as interfaces, which avoids local refinement around the fractures. The original problem is then transformed into a new one, namely the reduced fracture model, where the interaction between the fractures and the surrounding rock matrix is taken into account. For a general description of dimensional reduction for fracture modeling, we refer to [1–3, 10, 18, 27, 30, 35] and the references therein. We also mention [29, 39] where the full dimensional flow problem in fractured porous media is studied using either an upscaling method or the dual-porosity dual-permeability model.

In this paper, we are concerned with numerical algorithms for the reduced fracture model of linear advection-diffusion equations in a fractured porous medium in which the fracture has larger permeability than the surrounding porous media. In such a case, the physical processes in the fracture happen faster than those in the surrounding rock matrix. Thus, using a single time step size throughout the entire domain of calculation is computationally inefficient. In this work, we are interested in numerical schemes which allow local time stepping in the rock matrix and in the fracture as well as different time discretizations for the advective and diffusive processes. This can be achieved by combining global-in-time domain decomposition methods with operator splitting.

Global-in-time DD methods [21–24, 26] provide a powerful tool to perform parallel simulations of time-dependent physical phenomena with different time steps across the domain. The main idea is to first decouple the given dynamic system into dynamic subsystems defined on the subdomains (resulting from a spatial decomposition), then solve time-dependent problems in each subdomain at each iteration and exchange information over the space-time interfaces between the subdomains. Using substructuring techniques, the multidomain problem can be reduced into a space-time interface problem which is solved iteratively. There are basically two classes of global-in-time DD methods: Schur-type and Schwarz-type methods. The former is based on physical transmission conditions (e.g., the Dirichlet-Neumann and Neumann-Neumann waveform relaxation methods [8, 16, 17, 28, 31, 36]) while the latter is derived using more general transmission conditions, such as Robin or Ventcel conditions. These transmission conditions contain some free parameters which play a similar role as a preconditioner and can be optimized to improve the convergence of the method. This feature makes the second class related to the Optimized Schwarz Waveform Relaxation (OSWR) approach [5, 6, 11–13, 19, 20].

Both global-in-time DD classes were studied in [23] to handle the linear porous-media