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Iterative Pure Source Transfer Domain Decomposition Methods for Helmholtz Equations in Heterogeneous Media

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Abstract. We extend the pure source transfer domain decomposition method (PSTDDM) to solve the perfectly matched layer approximation of Helmholtz scattering problems in heterogeneous media. We first propose some new source transfer operators, and then introduce the layer-wise and block-wise PSTDDMs based on these operators. In particular, it is proved that the solution obtained by the layer-wise PSTDDM in $\mathbb{R}^2$ coincides with the exact solution to the heterogeneous Helmholtz problem in the computational domain. Second, we propose the iterative layer-wise and block-wise PSTDDMs, which are designed by simply iterating the PSTDDM alternatively over two staggered decompositions of the computational domain. Finally, extensive numerical tests in two and three dimensions show that, as the preconditioner for the GMRES method, the iterative PSTDDMs are more robust and efficient than PSTDDMs for solving heterogeneous Helmholtz problems.

AMS subject classifications: 65N55, 65F10, 65F08, 65Y05, 65Y20, 65N30, 78A40

Key words: Helmholtz equation, large wave number, perfectly matched layer, source transfer, domain decomposition method, preconditioner, heterogeneous problem.

1 Introduction

The purpose of this paper is to extend the pure source transfer domain decomposition method (PSTDDM) proposed in [17] to solve the Helmholtz scattering problem in heterogeneous media in $\mathbb{R}^d (d=2,3)$:

$$\Delta u+k^2u=f \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

$$\left|\frac{\partial u}{\partial r}-\mathbf{i}ku\right|=o\big(|x|^{\frac{1-d}{2}}\big) \quad \text{as } |x|\to\infty, \tag{1.2}$$

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where $k(x)=\frac{\omega}{c(x)}$ with $c(x)$ the wave speed, $\mathbf{i}$ denotes the imaginary unit, $c(x)-1$ and $f\in L^2(\mathbb{R}^d)$ are compactly supported on a bounded domain Ω. Here $\partial/\partial r$ denotes the derivative in radial direction $x/|x|$. The computational domain is assumed to be truncated by using the perfectly matched layer [5,9,10,12,30,31].

The Helmholtz problem (1.1)-(1.2) appears in various applications, such as acoustic, elastic, and electromagnetic scattering problems. Due to the highly indefinite nature of the Helmholtz problem with large wave number, it is challenging to solve the linear algebraic systems resulting from its numerical discretization, which usually contain huge numbers of degrees of freedom [2,3,15,28,29,46]. In particular for the cases of heterogeneous media, the stability of the Helmholtz problem on media with discontinuous, highly oscillating wave speed, may exponentially depend on the frequency ω (cf. [44]). Over the last few decades many discretization techniques for Helmholtz problems have been proposed and discussed in literature (see, e.g., [1,3,6,7,15,16,18,22,23,27,33,37–40,42,43,47,49]). And important research efforts have gone into developing efficient preconditioners for the Helmholtz equation. Engquist and Ying [19,20] introduced a new sweeping preconditioner for the iterative solution of the Helmholtz equation, which has linear application cost, and the preconditioned iterative solver converges in a number of iterations that is essentially independent of the number of unknowns or the frequency (see also [21,26,35,45]). Inspired by [20], a source transfer domain decomposition method (STDDM) [11] along with a rigorous convergence analysis is proposed, which can be used as an efficient preconditioner for the GMRES method for solving discrete Helmholtz equations truncated by the perfectly matched layer (PML) [4,11,33,48]. There are also many other methods, such as optimized Schwarz methods (see, e.g., [13,24,25,34,36,41]), multigrid methods [7,18], and some STDDM related methods [32,48]. The reader is referred to [26] for an overview of preconditioners for the Helmholtz Equation.

The PSTDDM proposed in [17] is a modification of the STDDM. After decomposing the domain into N non-overlapping layers, the STDDM is composed of two series steps of sources transfers and wave expansions, where $N-1$ truncated PML problems on two adjacent layers and $N-2$ truncated half-space PML problems are solved successively. The PSTDDM consists merely of two parallel source transfer steps in two opposite directions, and in each step $N-1$ truncated PML problems on two adjacent layers are solved successively. One benefit of such a modification is that the truncated PML problems on two adjacent layers can be further solved by the PSTDDM along directions parallel to the layers. And therefore, we obtain a block-wise PSTDDM, which reduces the size of subdomain problems and is more suitable for large-scale problems. For Helmholtz problems with constant wave number in $\mathbb{R}^2$, [17] has proved the exponential convergence of PSTDDMs for the truncated PML problem in bound domain and has shown numerically that they are robust and efficient as preconditioners for the GMRES method. However, PSTDDMs may not work for problems in high contrast media.

In this paper, we aim to extend the PSTDDM to solve heterogeneous Helmholtz equations. We first propose some new source transfer operators which have the same proper-