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Domain Decomposition Methods for Diffusion Problems with Discontinuous Coefficients Revisited

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Abstract. In this paper, we revisit some nonoverlapping domain decomposition methods for solving diffusion problems with discontinuous coefficients. We discover some interesting phenomena, that is, the Dirichlet-Neumann algorithm and Robin-Robin algorithms may make full use of the ratio of coefficients in some special cases. Detailedly, in the case of two subdomains, we find that their convergence rates are $\mathcal{O}(\nu_1/\nu_2)$ if $\nu_1 < \nu_2$, where ν_1,ν_2 are coefficients of two subdomains. Moreover, in the case of many subdomains with red-black partition, the condition number bounds of Dirichlet-Neumann algorithm and Robin-Robin algorithm are $1+\epsilon(1+\log(H/h))^2$ and $C+\epsilon(1+\log(H/h))^2$, respectively, where ϵ equals $\min\{\nu_R/\nu_B,\nu_B/\nu_R\}$ and ν_R,ν_B are the coefficients of red and black domains. By contrast, Neumann-Neumann algorithm and Dirichlet-Dirichlet algorithm could not obtain such good convergence results in these cases. Finally, numerical experiments are preformed to confirm our findings.

AMS subject classifications: 65N30, 65N55

Key words: Diffusion problem, discontinuous coefficients, finite elements, domain decomposition.

1 Introduction

Diffusion problem is a quite important model which is encountered in many physical problems and practical application fields. It is of great significance to solve diffusion equations numerically. One of the difficulties is that the diffusion coefficients are usually strongly discontinuous. A natural choice to overcome the difficulty is to use nonoverlapping domain decomposition (DD) methods to solve such kind of problems. Actually, there are lots of literature in the study of solving strong discontinuous problems

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by nonoverlapping DD methods. For instance, Mandel and Brezina [6] develop a balancing domain decomposition method for steady-state diffusion problem. In [4], a FETI algorithm is proposed and it is proved that the bounds on the rate of convergence are independent of possible jumps of the coefficients. In [8, 9], Sarkis design Schwarz preconditioners for discontinuous coefficients problems by using both conforming and nonconforming elements. In [3], a Robin-Robin preconditioner is proposed for advection-diffusion problems with discontinuous coefficients. For more study of this aspect, we refer to [7, 11] and the references cited therein.

We may find that the algorithms in most of the literature achieve convergence rates or condition number bounds independent of the jumps of coefficients. Wether there is a better result? For general cases, it could not be improved. However, we find a better result in some special cases, that is, the discontinuous coefficients may accelerate the convergence of Dirichlet-Neumann (D-N) algorithm and Robin-Robin (R-R) algorithm in the case of two subdomains and the case of many subdomains with red-black partition. Detailedly, if we suppose ν_1,ν_2 are the discontinuous coefficients in the case of two subdomains, then the convergence rates of the D-N algorithm and the R-R algorithm will completely depend on the ratio of the smaller coefficient to the larger coefficient, i.e. ν_1/ν_2 if $\nu_1<\nu_2$. Here we should clear that unlike the discontinuous coefficients case, the convergence rates of D-N algorithm and R-R algorithm are bounded by a constant which is independent of mesh size h and less than 1 strictly in the case ν_1 equals ν_2. In the case of many subdomains with red-black partition, the D-N algorithm and the R-R algorithm are always regarded as preconditioned methods and the corresponding condition number bounds are $1+\epsilon(1+\log(H/h))^2$ and $C+\epsilon(1+\log(H/h))^2$, respectively, where ϵ only depends on the ratio of the coefficients of red domains and black domains. Gander and Dubois [2] also find a similar phenomenon in the case of two symmetric subdomains. But they use the Fourier analysis to analyze it, as a result, their result is hard to extend to general cases. Meanwhile, we estimate the convergence rate by analyzing the spectra radium of error reduction operators and analyzing the condition numbers of preconditioned systems. Finally, all the results are confirmed by numerical experiments.

The paper is organized as follows: In Section 2, we introduce the model problem and domain decomposition methods. In Section 3, we analyze the influence of coefficients on convergence rates in the case of two subdomains with subdomains symmetric and nonsymmetric. In Section 4, the preconditioned systems in the case of many subdomains with red-black partition are described and the bounds on the condition numbers are given. Finally, we perform several numerical experiments to verify our conclusions.

2 Model problems and domain decomposition algorithms

We consider the following elliptic problem with discontinuous coefficients:

$$\begin{cases} -\nabla\cdot(\nu(\mathbf{x})\nabla u)=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$