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A New Upwind Finite Volume Element Method for Convection-Diffusion-Reaction Problems on Quadrilateral Meshes

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Abstract. This paper is devoted to constructing and analyzing a new upwind finite volume element method for anisotropic convection-diffusion-reaction problems on general quadrilateral meshes. We prove the coercivity, and establish the optimal error estimates in H^1 and L^2 norm respectively. The novelty is the discretization of convection term, which takes the two terms Taylor expansion. This scheme has not only optimal first-order accuracy in H^1 norm, but also optimal second-order accuracy in L^2 norm, both for dominant diffusion and dominant convection. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65N08, 65N12

Key words: Convection-diffusion-reaction, upwind finite volume method, coercivity, optimal convergence rate in L^2 norm.

1 Introduction

In this paper, we consider the anisotropic convection-diffusion-reaction equation

$$\begin{cases} -\nabla\cdot(\boldsymbol{\kappa}\nabla u-\mathbf{b}u)+cu=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a polygonal domain with boundary $\partial\Omega$, the symmetric diffusion tensor $\boldsymbol{\kappa}(x,y)=\big(\kappa_{ij}(x,y)\big)_{i,j=1,2}$ satisfies the uniformly elliptic condition on Ω, i.e., there exist constants $0<\mu_1<\mu_2$, such that

$$\mu_1|\xi|^2\le(\boldsymbol{\kappa}(x,y)\xi,\xi)\le\mu_2|\xi|^2,\quad \forall\xi\in\mathbb{R}^2, \tag{1.2}$$

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$\mathbf{b}(x,y)=(b_1(x,y),b_2(x,y))^T$ is convection velocity, reaction term function $c(x,y)\geq c_{\min}>0$, where $c_{\min}$ is a constant, and the source term $f\in L^2(\Omega)$.

Convection-diffusion-reaction equation appears in various fields of engineering and science, such as fluid mechanics, petroleum reservoir simulation, groundwater prediction, and environmental protection. Many methods have been used to solve convection-diffusion equations including the finite difference method, the finite element method, and the finite volume method. It is well known that the straightforward application of these methods to the case of dominant convection may lead to spurious oscillations in the numerical solution. To avoid this drawback, many techniques have been developed, for example, streamline upwind Petrov-Galerkin method [1], residual-free bubbles approach [2], variational multiscale method [3], and upwind technique [4].

Due to the local conservation, the finite volume methods are widely used in numerical simulation for partial differential equations. According to the position of the unknowns on the meshes, the finite volume methods can usually be divided into cell-centered type and vertex-centered type. Cell-centered type of finite volume methods can be viewed as an extension of finite difference methods [5, 6], the construction of which is simple and flexible. Finite volume element method (FVEM) is a kind of special Petrov-Galerkin method, which belongs to the vertex-centered type [7]. FVEM has not only low-order element schemes [8–14], but also high-order element schemes [15–22]. Many first-order finite volume methods are constructed using upwind technique for convection-diffusion equations [7, 23–30].

Many researchers focus on developing finite volume methods for convection-diffusion equations, which have high-order accuracy and avoid spurious oscillations. For the cell-centered finite volume methods, many schemes are constructed by using slope limiter [31–35], or two terms Taylor expansion technique [36,37]. In addition, a complete flux scheme (CFS) with second order accurate was developed on Cartesian grids with scalar diffusion [38]. Thereby, in order to deal with anisotropic diffusion tensors, Hanz Martin Cheng et al. establish a generalised complete flux scheme (GCFS) by combining CFS and the hybrid mimetic mixed method [39]. For FVEM, some schemes on rectangular meshes were proposed by constructing special dual meshes [40–42]. However, constructing special dual elements according to Peclet number needs additional computational cost. Under the standard dual partition, we hope to construct an upwind FVEM on general quadrilateral meshes with optimal convergence rate in L^2 norm.

In this paper, we construct a new upwind finite volume element method for the anisotropic convection-diffusion-reaction problems on general quadrilateral meshes. The trial space is taken as bilinear element space on quadrilateral meshes, and test space is taken as a piecewise constant function space on dual partition. We discrete the diffusion term by standard bilinear element finite volume method, which allows us to handle anisotropic diffusion tensors. Especially, the key ingredient is the discretization of the convection term, inspired by the idea in [37], we take the two terms of Taylor expansion of the solution at the upstream point to replace the solution on dual element boundary. Compared to the standard upwind scheme whose L^2 convergence rate is $\mathcal{O}(h)$, the