Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0232

Vol. **35**, No. 3, pp. 816-858
March 2024

Postprocessing Techniques of High-Order Galerkin Approximations to Nonlinear Second-Order Initial Value Problems with Applications to Wave Equations†

Mingzhu Zhang[1] and Lijun Yi[1,*]

[1] *Department of Mathematics, Shanghai Normal University, Shanghai 200234, China.*

Received 27 August 2023; Accepted (in revised version) 27 December 2023

Abstract. The aim of this paper is to propose and analyze two postprocessing techniques for improving the accuracy of the C^1- and C^0-continuous Galerkin (CG) time stepping methods for nonlinear second-order initial value problems, respectively. We first derive several optimal a priori error estimates and nodal superconvergent estimates for the C^1- and C^0-CG methods. Then we propose two simple but efficient local postprocessing techniques for the C^1- and C^0-CG methods, respectively. The key idea of the postprocessing techniques is to add a certain higher order generalized Jacobi polynomial of degree $k+1$ to the C^1- or C^0-CG approximation of degree k on each local time step. We prove that, for problems with regular solutions, such postprocessing techniques improve the global convergence rates for the L^2-, H^1- and L^∞-error estimates of the C^1- and C^0-CG methods with quasi-uniform meshes by one order. As applications, we apply the superconvergent postprocessing techniques to the C^1- and C^0-CG time discretization of nonlinear wave equations. Several numerical examples are presented to verify the theoretical results.

AMS subject classifications: 65L05, 65L60, 65L70, 65M60

Key words: Galerkin time stepping, second-order initial value problem, superconvergent postprocessing.

---

1 Introduction

The initial value problems (IVPs) of second-order ordinary differential equations (ODEs) play an important role in theoretical and numerical analysis of physical and engineering problems. Many important dynamical systems are governed by a system of second-order ODEs. Besides, many second-order time dependent problems, such as the nonlinear

†This paper is dedicated to Professor Benqi Guo on the occasion of his 80th Birthday.

*Corresponding author. *Email addresses:* niruochenxi@163.com (M. Zhang), ylj5152@shnu.edu.cn (L. Yi)

wave equations are often reduced to second-order ODEs after space semidiscretization. Traditional approaches for solving the IVPs of second-order ODEs generally rely on finite difference, Runge-Kutta and Newmark-type schemes; see, e.g., [5, 13–15, 21] and the references therein.

Galerkin-type time stepping methods for the numerical integration of ODEs can be traced back to the 1970s. In 1972, Hulme [17, 18] investigated the continuous Galerkin (CG) method for the numerical integration of nonlinear first-order IVPs. In 1981, Delfour et al. [9] developed another approach with discontinuous approximations, the so-called discontinuous Galerkin (DG) method for nonlinear first-order IVPs. For further studies on the a posteriori error estimation and adaptive error control of the CG and DG time stepping methods for first-order IVPs, we refer to [11, 12]. Moreover, Baccouch [3, 4] studied the superconvergence and a posteriori error estimation of the DG method for the numerical integration of nonlinear second-order IVPs. Recently, more flexible hp-versions of the CG and DG time stepping methods (with variable local step-sizes and local approximation degrees) were developed for the first- and second-order IVPs; see, e.g., [1, 19, 22, 26–28].

In the context of Galerkin-type methods, postprocessing techniques are attractive and efficient ways to improve the accuracy of an already obtained Galerkin approximation. Various postprocessing techniques have been developed for Galerkin spatial discretization of boundary value problems or initial-boundary value problems. However, to the best of our knowledge, there are only a few studies on the postprocessing techniques of Galerkin time stepping methods for IVPs. For example, Baccouch [2, 3] developed superconvergent postprocessing techniques of the DG time stepping methods for the nonlinear first- and second-order IVPs with regular solutions, and applied the superconvergence results to design asymptotically exact a posteriori estimates, where the second-order IVP was treated by transforming it into a first-order system. Recently, we developed a superconvergent postprocessing technique of the CG time stepping method for nonlinear first-order IVPs with smooth and singular solutions in [30]. It is worth pointing out that, such postprocessing techniques proposed in [2, 3, 30] can be regarded as a simple correction for the Galerkin solutions by adding an additional higher order Radau (resp. Lobatto) polynomial of degree $k+1$ to the existed DG (resp. CG) solutions of degree k on each local time step.

In this paper, we shall consider postprocessing techniques of the C^1- and C^0-CG time stepping methods for the nonlinear second-order IVP of the form

$$\begin{cases} u''(t)=f(t,u(t),u'(t)), \quad t\in(0,T], \\ u(0)=u_0, \quad u'(0)=u_1, \end{cases} \tag{1.1}$$

although the approaches and results carry over to systems of such ODEs. Here, $u:[0,T]\to\mathbb{R}$ denotes the unknown solution, $u_0,u_1\in\mathbb{R}$ denote the given initial values. We always assume that the function $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the following uniformly Lipschitz