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High Order Asymptotic Preserving and Well-Balanced Schemes for the Shallow Water Equations with Source Terms

Guanlan Huang[1], Sebastiano Boscarino[2] and Tao Xiong[3,*]

[1] *School of Mathematics and Statistics & Key Laboratory of Analytical Mathematics and Applications (Ministry of Education) & Fujian Key Laboratory of Analytical Mathematics and Applications (FJKLAMA) & Center for Applied Mathematics of Fujian Province (FJNU), Fujian Normal University, Fuzhou, Fujian 350117, China.*

[2] *Department of Mathematics and Computer Science, University of Catania, Catania 95125, Italy.*

[3] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China.*

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Abstract. In this study, we investigate the Shallow Water Equations incorporating source terms accounting for Manning friction and a non-flat bottom topology. Our primary focus is on developing and validating numerical schemes that serve a dual purpose: firstly, preserving all steady states within the model, and secondly, maintaining the late-time asymptotic behavior of solutions, which is governed by a diffusion equation and coincides with a long time and stiff friction limit.

Our proposed approach draws inspiration from a penalization technique adopted in *[Boscarino et al., SIAM Journal on Scientific Computing, 2014]*. By employing an additive implicit-explicit Runge-Kutta method, the scheme can ensure a correct asymptotic behavior for the limiting diffusion equation, without suffering from a parabolic-type time step restriction which often afflicts multiscale problems in the diffusive limit. Numerical experiments are performed to illustrate high order accuracy, asymptotic preserving, and asymptotically accurate properties of the designed schemes.

AMS subject classifications: 65M06, 76M20, 35L04, 35B40

Key words: Shallow water equations, Manning friction, asymptotic preserving, well-balanced, implicit-explicit Runge-Kutta, high order.

*Corresponding author. *Email addresses:* glhuang@fjnu.edu.cn (G. Huang), boscarino@dmi.unict.it (S. Boscarino), taoxiong@ustc.edu.cn (T. Xiong)

1 Introduction

The Shallow Water Equations (SWEs) are a set of equations derived from the more comprehensive Navier-Stokes equations, which govern the behavior of fluids. This hyperbolic system finds widespread application in modeling fluid dynamics in various natural environments, including channels, rivers, and oceans. The study of these equations carries significant practical implications, with numerous applications extending to crucial areas such as the prediction of storm surge levels, tides, and alterations in coastlines caused by events like hurricanes, among many others [29,35,47,48,50].

In recent years, the SWEs have become more and more important in atmospheric flows, especially with different space and time scales. In this paper, we consider the SWEs with Manning friction and a non-flat bottom topology, which can be written as

$$\begin{cases} \partial_t h + \nabla\cdot\mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla\cdot\left(\dfrac{\mathbf{q}\otimes\mathbf{q}}{h}\right) + \nabla\left(\dfrac{g}{2}h^2\right) = -gh\nabla b - \gamma\mathbf{q}, \end{cases} \tag{1.1}$$

where h is the depth of the water layer, $\mathbf{u}=\mathbf{q}/h$ is the flow velocity, which are defined on a time-space domain $(t,\mathbf{x})\in\mathbb{R}^+\times\Omega$; g is the gravitational constant, $b(\mathbf{x})$ is the bottom topography which is independent of time; $\otimes$ denotes the Kronecker product, and $\gamma\mathbf{q}$ models the bottom friction. Here $\gamma=\dfrac{gk^2|\mathbf{q}|}{h^\eta}$ with $|\mathbf{q}|$ being the L^2 norm of $\mathbf{q}$, and k is the Manning coefficient to determine the intensity of the friction exerted by the bottom of the water. The larger the k is, the greater the friction exerted by the water. η is a parameter which is usually taken to be $7/3$.

To investigate the *late-time* behavior of solutions to the system (1.1), and following the lines of [13], a behavior of h and $\mathbf{q}$ in a long time simulation and dominant friction is considered. Such an asymptotic behavior is governed by a diffusive regime. For the convenience of analysis, we introduce a small parameter ε in order to scale the time t and the friction coefficient k, as follows:

$$t\leftarrow\frac{\hat{t}}{\varepsilon},\quad k\leftarrow\frac{\hat{k}}{\varepsilon}. \tag{1.2}$$

Then the system (1.1) can be rewritten as

$$\begin{cases} \varepsilon\partial_{\hat{t}} h + \nabla\cdot\mathbf{q} = 0, \\ \varepsilon\partial_{\hat{t}} \mathbf{q} + \nabla\cdot\left(\dfrac{\mathbf{q}\otimes\mathbf{q}}{h}\right) + \nabla\left(\dfrac{g}{2}h^2\right) = -gh\nabla b - \dfrac{1}{\varepsilon^2}\widehat{\gamma}\mathbf{q}, \end{cases} \tag{1.3}$$

with $\widehat{\gamma}=\dfrac{g\hat{k}^2|\mathbf{q}|}{h^\eta}$. In [6] (and the references therein), it was shown that, in the modeling of shallow water problems when the friction is particularly strong and very much dominates over the advection effects, the asymptotic regime, i.e. $\varepsilon\to 0$, associated with (1.3) is