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Towards Preserving Geometric Properties of Landau-Lifshitz-Gilbert Equation Using Multistep Methods

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Abstract. In this paper, we investigate two fundamental geometric properties of the Landau-Lifshitz-Gilbert (LLG) equation, namely the preservation of magnetization magnitude and the Lyapunov structure, by using multistep methods. While the majority of current multistep methods for solving the LLG equation are based on two-step discrete schemes, our research specifically focuses on investigating more general multistep methods. Our proposed methods encompass a range of multistep discrete schemes that allow for achieving any desired order of accuracy in the temporal domain. In this highly general framework, we demonstrate that the magnitude of magnetization is preserved within an error of order $(p+2)$ in time when employing a $(p+1)$th-order multistep discrete scheme. Additionally, the Lyapunov structure is preserved with a first-order error of temporal step size. Finally, some numerical experiments are presented to validate the accuracy of the proposed multistep discrete schemes.

AMS subject classifications: 49Q15, 65M06, 65L06, 82D40
Key words: Geometric property, multistep methods, Landau-Lifshitz-Gilbert equation, computational micromagnetics.

1 Introduction

In micromagnetics, the Landau-Lifshitz-Gilbert (LLG) equation is an essential equation for describing magnetization vector field dynamics. Because of the significant nonlinearity of the LLG equation, analytically solving the LLG equation is impossible for general

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case. As a consequence, numerical simulation has emerged as the primary tool for studying the LLG equation, with numerous numerical approaches having been developed. These approaches include Runge-Kutta methods, Euler methods, the mid-point rule, the Gauss-Seidel projection method, and others.

Two of the most fundamental geometrical properties in the dynamics of micromagnetics are the constant magnitude of magnetization and the Lyapunov structure. The latter implies that the free energy is a decreasing function of time for constant external field. However, some standard numerical methods, such as the Euler methods and the Runge-Kutta methods, often corrupt the inherent geometrical properties within the LLG time evolution [6,7]. To our knowledge, the prevalent numerical methods for preserving the geometrical properties of magnetization dynamics are the mid-point rule and the projection method. The mid-point rule, which precisely conserves the constant magnitude of magnetization and reproduces the Lyapunov structure, is a highly utilized technique in micromagnetics [7,28]. While the mid-point rule can reproduce some geometric properties well, it is limited to second-order accuracy in time and requires solving a large discrete system of coupled nonlinear equations. For more references on the mid-point rule in micromagnetic dynamics, we recommend readers to refer to [8,10,11,14,18,19,21].

The projection method is also widely utilized in micromagnetics simulation, achieving constant magnetization magnitude by renormalizing the magnetization at the end of each algorithmic step [9]. Although the simple normalization operation brings challenges for theoretical analysis, it possesses significant scalability, making it easy to combine with other numerical methods. For example, by combining the Gauss-Seidel approach and the projection method, Wang, García-Cervera and E [26] developed an efficient and unconditionally stable scheme known as the Gauss-Seidel projection method for solving the Landau-Lifshitz (LL) equation. Based on the second-order Crank-Nicolson scheme, a Crank-Nicolson partially-updated projection scheme is proposed for the LL equation in [30]. This scheme achieves an optimal second-order convergence rate in both time and space. More references based on the projection method in micromagnetics can be found in [2,12,13,15,29].

In addition to the mid-point rule and the projection method, several other numerical methods aiming at preserving geometric properties of the LLG equation have been developed. These methods include the semianalytical scheme [24,25], the predictor-corrector scheme [24], the Cayley transform [1], the Lagrangian multiplier method [3], the pseudo-symplectic scheme [5], and others.

In this paper, we investigate the preservation of geometric properties of the LLG equation – the magnetization magnitude and the Lyapunov structure, by applying general multistep methods for the time stepping of the LLG equation. While most multistep methods currently used for solving the LLG equation are based on a two-step discrete scheme [16,20,22,27], our research aims to explore more general multistep methods that encompass various multistep methods and can achieve any desired order of accuracy in the temporal domain. In this highly general framework, we demonstrate that the magnitude of magnetization is preserved within an error of order $(p+2)$ in time when