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High Order Compact Hermite Reconstructions and Their Application in the Improved Two-Stage Fourth Order Time-Stepping Framework for Hyperbolic Problems: Two-Dimensional Case

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Abstract. The accuracy and efficiency of numerical methods are hot topics in computational fluid dynamics. In the previous work [J. Comput. Phys. 355 (2018) 385] of Du et al., a two-stage fourth order (S_2O_4) numerical scheme for hyperbolic conservation laws is proposed, which is based on dimension-by-dimensional HWENO5 and WENO5 reconstructions and GRP solver, and uses a S_2O_4 time-stepping framework. In this paper, we aim to design a new type of S_2O_4 finite volume scheme, to further improve the compactness and efficiency of the numerical scheme. We design an improved S_2O_4 framework for two-dimensional compressible Euler equations, and develop nonlinear compact Hermite reconstructions to avoid oscillations near discontinuities. The new two-stage fourth order numerical schemes based on the above nonlinear compact Hermite reconstructions and GRP solver are high-order, stable, compact, efficient and essentially non-oscillatory. In addition, the above reconstructions and the corresponding numerical schemes are extended to eighth-order accuracy in space, and can be theoretically extended to any even-order accuracy. Finally, we present a large number of numerical examples to verify the excellent performance of the designed numerical schemes.

AMS subject classifications: 65M08, 65M12, 35L65, 76M12

Key words: Hyperbolic conservation laws, two-stage fourth order time-stepping framework, compact Hermite reconstruction, high order, GRP solver.

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1 Introduction

Fluid dynamics plays a crucial role in predicting fluid flow in various fields, such as aerospace engineering, weather forecasting, and more. Among these, the two-dimensional Euler system, which belongs to the class of hyperbolic conservation laws, serves as a classic model. The demand for precise prediction and simulation in these applications has driven the need for high accurate schemes. However, the pursuit of accuracy must be balanced with computational efficiency, making this an interesting and challenging problem. Additionally, in order to capture the details in the flow field more accurately, such as turbulence and shock waves, we need to develop high-resolution schemes, which can be achieved by constructing compact schemes. At the same time, compactness can effectively reduce the communication overhead between nodes, making the scheme more efficient in parallel computing.

For hyperbolic conservation laws, plenty of numerical methods have been developed, including the finite difference method (FDM), finite volume method (FVM), and finite element method (FEM). Specifically, in the realm of FVM, numerous high order schemes such as the essentially non-oscillatory (ENO) [1], weighted ENO (WENO) [2,3], and Hermite WENO (HWENO) [4, 5] schemes have emerged over the past decades. The ENO scheme avoids oscillations by selecting the smoothest stencil. However, this approach disregards valuable data from other stencils. As an advancement, the WENO scheme accepts all stencils in smooth regions to attain higher order accuracy, while concurrently preserving the capacity to avoid oscillations. Subsequently, Qiu and Shu introduced the HWENO scheme. In this scheme, each cell contains an increased degree of freedom, encompassing not only the cell average but also the gradient average. This leads to a more narrow stencil for the HWENO scheme, making it more compact compared to the traditional WENO scheme.

Many of the above schemes adopted the strong-stability preserving Runge-Kutta (SSP-RK) time-stepping methods [6], a multi-stage time-stepping method, such as the third order SSP-RK method, which can achieve third order in time through three time-stages. This method can be expressed as a convex combination of forward Euler method, which implies many nice properties that can be generalized. However, the numerous intermediate time-stages make the resulting scheme less compact. Alternatively, there is a class of time-stepping method known as the Lax-Wendroff type methods [1,7,8], which can achieve high order accuracy in time through one time-stage. The resulting schemes are more compact, but their formulation and coding could be rather complicated, especially for multidimensional systems. Afterwards, a class of multi-stage multi-derivative (MSMD) time-stepping methods [9,10] emerged, which balance the complexity and compactness. Among them, the S_2O_4 framework makes a certain contribution, with many related works [11, 12]. Note that the following S_2O_4 framework here specifically refers to the time-stepping framework proposed in [11], using a Lax-Wendroff type flow solver and a two-moment reconstruction, which is different from the previous one in [9]. Two-