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Optimal Error Estimates of a Discontinuous Galerkin Method for Stochastic Allen-Cahn Equation Driven by Multiplicative Noise

Xu Yang[1], Weidong Zhao[2] and Wenju Zhao[2,*]

[1] *School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, China.*
[2] *School of Mathematics, Shandong University, Jinan, Shandong 250100, China.*

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Abstract. In this paper, we develop and analyze an efficient discontinuous Galerkin method for stochastic Allen-Cahn equation driven by multiplicative noise. The proposed method is realized by symmetric interior penalty discontinuous Galerkin finite element method for space domain and implicit Euler method for time domain. Several new estimates and techniques are developed. Under some suitable regularity assumptions, we rigorously establish strong convergence results for the proposed fully discrete numerical scheme and obtain optimal convergence rates in both space and time. Numerical experiments are also carried out to validate our theoretical results and demonstrate the effectiveness of the proposed method.

AMS subject classifications: 60H15, 60H35, 65C30, 65M60

Key words: Stochastic Allen-Cahn equation, strong convergence, discontinuous Galerkin method, variational solution, multiplicative noise.

1 Introduction

In this paper, we consider strong approximation for the Itô type Allen-Cahn equation with multiplicative noise of the form

$$\mathrm{d}u+Au\,\mathrm{d}t=f(u)\,\mathrm{d}t+g(u)\,\mathrm{d}W \quad \text{in } D\times(0,T], \tag{1.1}$$

$$\frac{\partial u}{\partial \mathbf{n}}=0 \quad \text{on } \partial D\times[0,T], \tag{1.2}$$

$$u=u_0 \quad \text{in } D\times\{t=0\}. \tag{1.3}$$

*Corresponding author. *Email addresses:* `xuyang96@cumt.edu.cn` (X. Yang), `wdzhao@sdu.edu.cn` (W. Zhao), `zhaowj@sdu.edu.cn` (W. Zhao)

Here T is a fixed positive time instant, D is a bounded convex domain in $\mathbb{R}^d$, $d=1,2,3$, with polygonal boundary ∂D, $\mathbf{n}$ denotes the unit outward normal vector to the boundary ∂D, the operator A denotes the linear elliptic operator defined by $Au:=-\Delta u$, $f\colon \mathbb{R}\to\mathbb{R}$ is given by $f(u):=\frac{1}{\epsilon^2}(u-u^3)$ with parameter ϵ the interfacial width, $g\colon \mathbb{R}\to\mathbb{R}$ is an appropriate regular function which will be specified in the next section, and the driving noise W is a standard Wiener process defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with normal filtration $\mathbb{F}=\{\mathcal{F}_t\}_{0\le t\le T}$. For the sake of notational simplicity, we focus our discussion on the one dimensional noise case and it is straightforward to generalize the analysis below to the multidimensional noise cases.

The deterministic Allen-Cahn equation was first introduced by Allen and Cahn in [1] to describe the motion of anti-phase boundaries of a binary alloy at a fixed temperature. Since then, as a fundamental tool, Allen-Cahn equation plays an important role in many complicated moving interface problems such as material science, fluid dynamics, image analysis and mean curvature flow [27]. Frequently, due to the presence of external perturbations, lack of information, uncertainty in the measurements, and incomplete knowledge of certain physical parameters, a stochastic noise is usually added in the model to make the description of the system more realistic, which results in the stochastic Allen-Cahn equation (SAC). Note that sharing the same structure with the deterministic case, the model (1.1) admits superlinearly growing coefficient and in general it can not be solved explicitly. In recent years, the construction and numerical analysis of efficient approximation schemes for stochastic Allen-Cahn equation have begun to attract the attention of many researchers and plenty of interesting numerical methods have been developed, analyzed and tested, see e.g. [3–5,9,17,28,29,37,38,41,45] and the references therein. However, most of the aforementioned works have considered the additive noise case, where the stochastic convolution plays a key role in the numerical analysis. In comparison with large amounts of numerical studies of stochastic Allen-Cahn equation with additive noise, the numerical studies of multiplicative noise driving stochastic Allen-Cahn equation are still very limited (see e.g. [21,30,37,39]).

Discontinuous Galerkin (DG) methods are a class of finite element methods with the basis functions which can be completely discontinuous [16]. DG methods are eligible for high order schemes in space and flexible of parallel implementation of handling the complex problem, which have been widely used to solve deterministic partial differential equations (PDEs). And we refer to the books [16, 18, 19, 42] for more details on the development of DG methods in all aspects including algorithm design, analysis, implementation and applications. As far as the stochastic cases are concerned, DG methods for stochastic partial differential equations (SPDEs) can inherit the advantages of their counterparts for deterministic PDEs and they are good choices to solve SPDEs with sharp feature. For this reason, recently, many researchers turn their attentions to working in the design and analysis of DG methods for SPDEs. A symmetric interior penalty DG and local DG for SPDEs are proposed and investigated in [34,46]. Discontinuous Galerkin methods for stochastic wave equations, Helmholtz equation, conservation laws and Kdv equations are investigated in [2,10,11,26,32,33]. Mean-square convergence analysis of a symplectic