Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0233

Vol. **36**, No. 1, pp. 221-247
July 2024

The Lowest-Order Stabilized Virtual Element Method for the Stokes Problem

Xin Liu[1,2], Qixuan Song[1], Yu Gao[3] and Zhangxin Chen[4]*

[1] *School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, 710129, China.*

[2] *MOE Key Laboratory for Complexity Science in Aerospace, Northwestern Polytechnical University, Xi'an, 710129, China.*

[3] *Department of Basic Courses, Shaanxi Railway Institute, Weinan, 714000, China.*

[4] *Department of Chemical and Petroleum Engineering, Schulich School of Engineering, University of Calgary, 2500 University Drive N.W., Calgary, Alberta T2N 1N4, Canada.*

Received 28 August 2023; Accepted (in revised version) 29 March 2024

Abstract. In this paper, we develop and analyze two stabilized mixed virtual element schemes for the Stokes problem based on the lowest-order velocity-pressure pairs (i.e., a piecewise constant approximation for pressure and an approximation with an accuracy order $k=1$ for velocity). By applying local pressure jump and projection stabilization, we ensure the well-posedness of our discrete schemes and obtain the corresponding optimal H^1- and L^2-error estimates. The proposed schemes offer a number of attractive computational properties, such as, the use of polygonal/polyhedral meshes (including non-convex and degenerate elements), yielding a symmetric linear system that involves neither the calculations of higher-order derivatives nor additional coupling terms, and being parameter-free in the local pressure projection stabilization. Finally, we present the matrix implementations of the essential ingredients of our stabilized virtual element methods and investigate two- and three-dimensional numerical experiments for incompressible flow to show the performance of these numerical schemes.

AMS subject classifications: 65N30, 65N12, 65N15, 76D05

Key words: Stokes equations, stabilized virtual element scheme, pressure jump, pressure projection, polygonal meshes.

*Corresponding author. *Email addresses:* `zhachen@ucalgary.ca` (Z. Chen), `liuxinjy@gmail.com` (X. Liu), `qixuan_song@mail.nwpu.edu.cn` (Q. Song), `gy19890524@163.com` (Y. Gao)

 ©2024 Global-Science Press

1 Introduction

Incompressible Stokes flow, as one of the most important and valuable problems, involves many practical applications, such as oil exploration, pipeline transportation, sedimentation, modeling of bio-suspensions, construction of efficient fibrous filters, and development of energy efficient micro-fluidic devices. Due to the limitations of fluid experiments, using computer-based numerical simulation remains an effective and flexible method in practical applications. The classical finite element method, especially with the lowest-order conforming pair (i.e., piecewise linear/bilinear C^0 velocities and piecewise constant pressures) with convenient construction (i.e., simpler shape functions) and fast implementation (i.e., fewer degrees of freedom), has become the preferred solution for such problems; see, e.g., [1–4] and the references cited therein. However, an important fact is that the lowest-order velocity-pressure pairs violate the LBB [5] (inf-sup) stability condition, which often leads to unphysical pressure oscillations. To overcome this difficulty, a series of methods have been developed, such as penalty methods [6–8], consistently stabilized methods [9, 10], pressure gradient projection methods [11–13], related local pressure gradient stabilization methods [14], offset pressure stabilization methods [15], and projection-based stabilized methods [16, 17], among others.

As an extension of the classical finite elements to general polygonal elements, the virtual element method (VEM) has gained widespread attention since its theory [18] and matrix implementation [19] were proposed. Then the authors in [20] enhanced a discrete space and gave a specific process in calculating an L^2-projection operator for a three-dimensional reaction-diffusion problem. By combing the ideas of VEM with other methods, the H^α-conforming VEM [21, 22], the nonconforming VEM [23, 24], and the $H(\text{div})/H(\text{curl})$-VEM [25, 26] were designed. Due to the advantages of the virtual element method in mesh flexibility and structure-preserving spatial construction, the VEM has been widely used in adaptive mesh refinement [27], elliptic bulk-surface PDEs [28], structural mechanics elasticity [29, 30] and incompressible fluid problems [31–36].

Combining the widespread practical applications of the lowest-order elements with the advantages of the virtual element method, it is crucial to construct the lowest-order virtual element pair, which, in fact, not only faces a similar situation to the lowest-order mixed finite elements (that is, the pair fails to satisfy the inf-sup stability condition), but also needs to consider the computability of additional stabilization terms introduced to meet this stability condition (since the VEM lacks explicit expressions of basis functions). About these challenges, the authors in [37] have developed the 'equal-order' stabilized virtual element pairs for the Stokes problem on polygonal meshes, utilizing a projection-based stabilization to circumvent the discrete inf-sup condition. In addition, the authors in [38] have proposed a least-squares type stabilization VEM for the Stokes problem, which is suitable for arbitrary combinations of velocity and pressure. Also, there is some research on stabilized virtual element methods for other problems, such as the Navier-Stokes [39], Oseen [40], advection-diffusion-reaction [41–43] problems, among others. Furthermore, it is worth mentioning that [31] has also provided a lowest-order virtual el-