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Quinpi: Integrating Stiff Hyperbolic Systems with Implicit High Order Finite Volume Schemes

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Abstract. Many interesting physical problems described by systems of hyperbolic conservation laws are stiff, and thus impose a very small time-step because of the restrictive CFL stability condition. In this case, one can exploit the superior stability properties of implicit time integration which allows to choose the time-step only from accuracy requirements, and thus avoid the use of small time-steps. We discuss an efficient framework to devise high order implicit schemes for stiff hyperbolic systems without tailoring it to a specific problem. The nonlinearity of high order schemes, due to space- and time-limiting procedures which control nonphysical oscillations, makes the implicit time integration difficult, e.g. because the discrete system is nonlinear also on linear problems. This nonlinearity of the scheme is circumvented as proposed in (Puppo et al., Comm. Appl. Math. & Comput., 2023) for scalar conservation laws, where a first order implicit predictor is computed to freeze the nonlinear coefficients of the essentially non-oscillatory space reconstruction, and also to assist limiting in time. In addition, we propose a novel conservative flux-centered a-posteriori time-limiting procedure using numerical entropy indicators to detect troubled cells. The numerical tests involve classical and artificially devised stiff problems using the Euler's system of gas-dynamics.

AMS subject classifications: 65M08, 65M20, 35L65, 65L04
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1 Introduction

Mathematical models for the description of fluids, plasmas, and many other physical phenomena, are typically given in terms of systems of hyperbolic conservation laws.

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These systems are characterized by a set of multi-dimensional partial differential equations (PDEs) that express the conservation of various physical quantities in terms of their respective fluxes. A prototypical example is provided by the Euler's equations for gas-dynamics describing the conservation of mass, momentum, and energy of a gas.

In this work we focus on one-dimensional systems of $m \geq 1$ hyperbolic conservation laws:

$$\frac{\partial}{\partial t}\mathbf{u}(x,t)+\frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}(x,t))=\mathbf{0}, \tag{1.1}$$

where, $\mathbf{u}:\mathbb{R}\times\mathbb{R}_0^+\to\mathbb{R}^m$ is the quantity of interest, and $\mathbf{f}:\mathbb{R}^m\to\mathbb{R}^m$ is the vector of the flux functions. System (1.1) is hyperbolic when the eigenvalues $\{\lambda_j(\mathbf{u}(x,t))\}_{j=1}^m$ of the associated Jacobian matrix are real and determine a complete set of eigenvectors. The eigenvalues of (1.1) provide the characteristic velocities, which describe the propagation speed of waves in the system. These waves can be either acoustic waves (shocks and rarefactions) or material waves (contact discontinuities). Requiring that the eigenvalues are real implies that the propagation speed of information through the system is finite.

Solving hyperbolic systems of conservation laws is a challenging task, both analytically and numerically, e.g. due to the occurrence of singularities or the need of devising high order accurate non-oscillatory methods to avoid low-resolution approximations. Another source of numerical difficulty is represented by stiff problems that occur when the system is characterized by speeds spanning different orders of magnitude, namely when $\frac{\max_{j=1,\cdots,m}|\lambda_j(\mathbf{u})|}{\min_{j=1,\cdots,m}|\lambda_j(\mathbf{u})|}\gg 1$. This happens, for instance, in gas-dynamics when the fluid speed is much less than the speed of the acoustic waves. In many applications the phenomenon of interest travels with a low speed. An example is provided by low-Mach number problems occurring when the equations governing the flow become stiff due to the very low fluid velocity compared to the speed of sound in the fluid. In these situations, the compressibility effects of the fluid can be neglected, and the fluid is almost incompressible. Then, if the interest is on the movement of the fluid, accuracy in the propagation of sound waves becomes irrelevant. For low-Mach problems we refer to [1,11,21–23,49].

Numerical schemes used to solve hyperbolic problems need to be carefully designed to handle the stiff regime. In fact, it is well-known that explicit schemes are subject to the Courant-Friedrichs-Levy (CFL) stability condition that specifies a constraint on the numerical speed in relation to the maximum speed of information propagating in the system. More precisely, let Δt and h be the time-step and the mesh width of a numerical scheme, respectively. We define the *numerical speed* as $s_n = {}^{h}/_{\Delta t}$ which approximates the speed with which the numerical data propagate in the discretized system. Then, the CFL condition imposes that s_n must be faster than the maximum speed of propagation to ensure that information does not travel too far between adjacent space cells during one time-step. For this reason, the stability request on the time-step of explicit schemes becomes very restrictive for stiff problems due to the presence of fast waves, thus limiting the computational efficiency of the scheme. In contrast, implicit schemes can have supe-